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Recursion relations, generating functions, and unitarity sums in $\mathcal{N} = 4$ SYM theory

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Abstract: We prove that the MHV vertex expansion is valid for any NMHV tree amplitude of $\mathcal{N} = 4$ SYM. The proof uses induction to show that there always exists a complex deformation of three external momenta such that the amplitude falls off at least as fast as $1/z$ for large z. This validates the generating function for n-point NMHV tree amplitudes. We also develop generating functions for anti-MHV and anti-NMHV amplitudes. As an application, we use these generating functions to evaluate several examples of intermediate state sums on unitarity cuts of 1-, 2-, 3- and 4-loop amplitudes. In a separate analysis, we extend the recent results of arXiv:0808.0504 to prove that there exists a valid 2-line shift for any n-point tree amplitude of $\mathcal{N} = 4$ SYM. This implies that there is a BCFW recursion relation for any tree amplitude of the theory.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry

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1 Introduction

Recursion relations for tree amplitudes based on the original constructions of CSW [1] and BCFW [2, 3] have had many applications to QCD, $\mathcal{N} = 4$ SYM theory, general relativity, and $\mathcal{N} = 8$ supergravity.[1] In this paper we are concerned with recursion relations for n-point tree amplitudes $A_n(1, 2, \ldots, n)$ in which the external particles can be any set of gluons, gluinos, and scalars of $\mathcal{N} = 4$ SYM theory.

Recursion relations follow from the analyticity and pole factorization of tree amplitudes in a complex variable z associated with a deformation or shift of the external momenta. A valid recursion relation requires that the shifted amplitude vanishes as $z \to \infty$. This has been proven for external gluons by several interesting techniques, [1, 3, 5], but there is only partial information for amplitudes involving other types of particles [6]. Of particular relevance to us is a very recent result of Cheung [7] who shows that there always exists at least one valid 2-line shift for any amplitude of the $\mathcal{N} = 4$ theory in which one particle is a negative helicity gluon and the other $n - 1$ particles are arbitrary. We use SUSY Ward identities to extend the result to include amplitudes with n particles of any[2] type. Thus for $\mathcal{N} = 4$ SYM amplitudes there always exists a valid 2-line shift which leads to a recursion relation of the BCFW type. It is simplest for MHV amplitudes but provides a correct representation of all amplitudes.

For NMHV amplitudes the MHV vertex expansion of CSW is usually preferred, and this is our main focus. The MHV vertex expansion is associated with a 3-line shift [8], and it is again required to show that amplitudes vanish at large z under such a shift. To prove this, we use the BCFW representation to study n-point NMHV amplitudes in which one particle is a negative helicity gluon but other particles are arbitrary. Using induction on n we show that there is always at least one 3-line shift for which these amplitudes vanish in the large z limit. The restriction that one particle is a negative helicity gluon can then be removed using SUSY Ward identities. Thus there is a valid (and unique, as we argue) MHV vertex expansion for any NMHV amplitude of the $\mathcal{N} = 4$ theory.

Next we turn our attention to the generating functions which have been devised to determine the dependence of amplitudes on external states of the theory. The original and simplest case is the MHV generating function of Nair [9]. A very useful extension to diagrams of the CSW expansion for NMHV amplitudes was proposed by Georgiou, Glover, and Khoze [10]. MHV and NMHV generating functions were further studied in a recent paper [11] involving two of the present authors. A 1:1 correspondence was established

[1] Readers are referred to the review [4] and the references listed there.

[2] With the exception of one 4-scalar amplitude. See section 7.

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between the particles of $\mathcal{N}=4$ SYM theory and differential operators involving the Grassmann variables of the generating function. MHV amplitudes are obtained by applying products of these differential operators of total order 8 to the generating function, and an NMHV amplitude is obtained by applying a product of operators of total order 12. We review these constructions and emphasize that the NMHV generating function has the property that every 12th order differential operator projects out the correct MHV vertex expansion of the corresponding amplitude, specifically the expansion which is validated by the study of the large z behavior of 3-line shifts described above. In this sense the NMHV generating function is universal in $\mathcal{N}=4$ SYM theory. Its form does not contain any reference to a shift, but every amplitude is produced in the expansion which was established using a valid 3-line shift.

In [11] it was shown in examples at the MHV level how the generating function formalism automates and simplifies the sum over intermediate helicity states required to compute the unitarity cuts of loop diagrams. In this paper we show how Grassmann integration further simplifies and extends the MHV level helicity sums. We then apply the universal generating function to examples of helicity sums involving MHV and NMHV amplitudes.

Even in the computation of MHV amplitudes at low loop order, N^{2}MHV and N^{3}MHV tree amplitudes are sometimes required to complete the sums over intermediate states. We derive generating functions for all N^{k}MHV amplitudes in [12] (see also [13]). Note though that when the amplitudes have $k+4$ external lines these are equivalent to anti-MHV or anti-NMHV amplitudes. In this paper we discuss a general procedure to convert the conjugate of any N^{k}MHV generating function into an anti-N^{k}MHV generating function which can be used to compute spin sums. We study the n-point anti-MHV and anti-NMHV cases in detail and apply them in several examples of helicity sums. These include 3-loop and 4-loop cases.

We use conventions and notation given in appendix A of [11].

2 $\mathcal{N}=4$ SUSY Ward identities

The bosons and fermions of $\mathcal{N}=4$ SYM theory can be described by the following annihilation operators, which are listed in order of descending helicity:

$$B_+(i)\,, \qquad F_+^a(i)\,, \qquad B^{ab}(i)=\frac{1}{2}\,\epsilon^{abcd}B_{cd}(i)\,, \qquad F_a^-(i)\,, \qquad B^-(i)\,. \tag{2.1}$$

The argument i is shorthand for the 4-momentum p_i^μ carried by the particle. Particles of opposite helicity transform in conjugate representations of the SU(4) global symmetry group (with indices $a,b,\dots$), and scalars satisfy the indicated SU(4) self-duality condition. In this paper it is convenient to "dualize" the lower indices of positive helicity annihilators and introduce a notation in which all particles carry upper SU(4) indices, namely:

$$\begin{aligned} A(i) &= B_+(i)\,, & A^a(i) &= F_+^a(i)\,, & A^{ab}(i) &= B^{ab}(i)\,, \\ A^{abc}(i) &= \epsilon^{abcd}F_d^-(i)\,, & A^{abcd}(i) &= \epsilon^{abcd}B^-(i)\,. \end{aligned} \tag{2.2}$$

Note that the helicity (hence bose-fermi statistics) of any particle is then determined by the SU(4) tensor rank r of the operator $A^{a_1\cdots a_r}(i)$.

Chiral supercharges $Q^a \equiv -\epsilon^\alpha\, Q^a_\alpha$ and $\tilde{Q}_a \equiv \tilde{\epsilon}_{\dot\alpha}\tilde{Q}^{\dot\alpha}_a$ are defined to include contraction with the anti-commuting parameters $\epsilon^\alpha, \tilde{\epsilon}_{\dot\alpha}$ of SUSY transformations. The commutators of the operators Q^a and $\tilde{Q}_a$ with the various annihilators are given by:

$$\begin{aligned}
\big[\tilde{Q}_a, A(i)\big] &= 0\,, & \big[Q^a, A(i)\big] &= [i\,\epsilon]\, A^a(i)\,,\\
\big[\tilde{Q}_a, A^b(i)\big] &= \langle \epsilon\, i\rangle\, \delta^b_a\, A(i)\,, & \big[Q^a, A^b(i)\big] &= [i\,\epsilon]\, A^{ab}(i)\,,\\
\big[\tilde{Q}_a, A^{bc}(p)\big] &= \langle \epsilon\, i\rangle\, 2!\, \delta^{[b}_a\, A^{c]}(i)\,, & \big[Q^a, A^{bc}(i)\big] &= [i\,\epsilon]\, A^{abc}(i)\,,\\
\big[\tilde{Q}_a, A^{bcd}(i)\big] &= \langle \epsilon\, i\rangle\, 3!\, \delta^{[b}_a A^{cd]}(i)\,, & \big[Q^a, A^{bcd}(i)\big] &= [i\,\epsilon]\, A^{abcd}(i)\,,\\
\big[\tilde{Q}_a, A^{bcde}(i)\big] &= \langle \epsilon\, i\rangle\, 4!\, \delta^{[b}_a A^{cde]}(i)\,, & \big[Q^a, A^{bcde}(i)\big] &= 0\,.
\end{aligned} \tag{2.3}$$

Note that $\tilde{Q}_a$ raises the helicity of all operators and involves the spinor angle brackets $\langle \epsilon\, i\rangle$. Similarly, Q^a lowers the helicity and spinor square brackets $[i\,\epsilon]$ appear.

It is frequently useful to suppress indices and simply use $\mathcal{O}(i)$ for any annihilation operator from the set in (2.2). A generic n-point amplitude may then be denoted by

$$A_n(1,2,\ldots,n) = \big\langle \mathcal{O}(1)\mathcal{O}(2)\ldots\mathcal{O}(n)\big\rangle\,. \tag{2.4}$$

SU(4) invariance requires that the total number of (suppressed) indices is a multiple of 4, i.e. $\sum_{i=1}^n r_i = 4m$.

It is well known, however, that amplitudes A_n with $n \geq 4$ vanish if $\sum_{i=1}^n r_i = 4$. To see this we use SUSY Ward identities, as in the particular case

$$\begin{aligned}
0 &= \big\langle \big[\tilde{Q}_1\,,\, A^1(1)A^{1234}(2)A(3)\ldots A(n)\big]\big\rangle\\
&= \langle \epsilon\, 1\rangle\big\langle A(1)A^{1234}(2)A(3)\ldots A(n)\big\rangle \,+\, \langle \epsilon\, 2\rangle\big\langle A^1(1)A^{234}(2)A(3)\ldots A(n)\big\rangle\,.
\end{aligned} \tag{2.5}$$

There are exactly two terms in the Ward identity. One can choose $|\epsilon\rangle \sim |2\rangle$ and learn that the first amplitude, involving one negative helicity and $n-1$ positive helicity gluons, vanishes. The second fermion pair amplitude must then also vanish.[3] We have chosen one specific example for clarity, but the argument applies to all amplitudes with $\sum_{i=1}^n r_i = 4$ and $n \geq 4$. To see this consider

$$0 \,=\, \big\langle \big[\tilde{Q}_1\,,\, \mathcal{O}(1)\mathcal{O}(2)\ldots\mathcal{O}(n)\big]\big\rangle\,. \tag{2.6}$$

SU(4) symmetry requires that the upper index 1 appears exactly twice among the operators $\mathcal{O}(i)$ and that the indices 2,3,4 each appear once. The commutator again contains two terms, one from each $\mathcal{O}(i)$ that carries the index 1. The argument above then applies immediately.

Let's continue and discuss the Ward identity (2.6) for the general case $\sum_{i=1}^n r_i = 4m,\ m \geq 2$. The upper index 1 must appear $m+1$ times among the $\mathcal{O}(i)$ and the indices

[3] This argument does not apply to the case $n = 3$ because the strong constraint of momentum conservation forces either $|1\rangle = c|2\rangle$ or $|1] = c|2]$. If the first occurs, then one cannot choose $|\epsilon\rangle$ so as to isolate one of the two terms in (2.5), and $A_3(1,2,3)$ with $\sum_{i=1}^n r_i = 4$ need not vanish for complex momenta. This amplitude is anti-MHV.

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2,3,4 each appear m times. The commutator then contains $m+1$ terms, and each of these involves an amplitude with $\sum_{i=1}^{n} r_i = 4m$.

Ward identities with the conjugate supercharges Q^a have the similar structure

$$0 = \left\langle \left[Q^1 \, , \mathcal{O}(1)\mathcal{O}(2)\ldots\mathcal{O}(n) \right] \right\rangle . \tag{2.7}$$

This is a non-trivial identity if the index 1 appears $m-1$ times, and the indices 2,3,4 each appear m times. The commutator then contains $n-m+1$ terms, each again with an amplitude with $\sum_{i=1}^{n} r_i = 4m$. To summarize, all amplitudes related by any one SUSY Ward identity must have the same total number of upper SU(4) indices. It is then easy to see the case $m=2$ corresponds to MHV amplitudes, $m=3$ to NMHV, while general N^{k}MHV amplitudes must carry a total of $4(k+2)$ upper indices.

In [11] a 1:1 correspondence between annihilation operators in (2.2) and differential operators involving the Grassmann variables η_{ia} of generating functions was introduced. We will need this correspondence in section 4 below, so we restate it here:

$$A(i) \leftrightarrow 1 \, , \qquad A^a(i) \leftrightarrow \frac{\partial}{\partial\eta_{ia}} \, , \qquad A^{ab}(i) \leftrightarrow \frac{\partial^2}{\partial\eta_{ia}\partial\eta_{ib}} \, , \tag{2.8}$$

$$A^{abc}(i) \leftrightarrow \frac{\partial^3}{\partial\eta_{ia}\partial\eta_{ib}\partial\eta_{ic}} \, , \qquad A^{abcd}(i) \leftrightarrow \frac{\partial^4}{\partial\eta_{ia}\partial\eta_{ib}\partial\eta_{ic}\partial\eta_{id}} \, .$$

Thus a particle state whose upper SU(4) rank is r corresponds to a differential operator of order r. In accord with [11] we will refer to the rank r as the η-count of the particle state. We showed in [11] that an MHV amplitude containing a given set of external particles can be obtained by applying a product of the corresponding differential operators of total order 8 to the MHV generating function, and NMHV amplitudes that are obtained by applying products of total order 12 to the NMHV generating function. The classification of amplitudes based on the total η-count of the particles they contain is a consequence of SU(4) invariance.

3 Valid 3-line shifts for NMHV amplitudes

The major goal of this section is to prove that there is at least one 3-line shift for *any* NMHV amplitude $A_n(m_1,\ldots,m_2,\ldots,m_3,\ldots)$ under which the amplitude vanishes at the rate $1/z$ or faster as $z\to\infty$. We show that this is true when the shifted lines m_1, m_2, m_3 share at least one common SU(4) index, and that such a shift is always available. The first step in the proof is to show that there is a valid 3-line shift for any NMHV amplitude $A_n(1^-,\ldots,m_2,\ldots,m_3,\ldots,n)$, with particle 1 a negative helicity gluon, m_2 and m_3 sharing a common SU(4) index, and the other states arbitrary. This requires an intricate inductive argument which we outline here and explain in further detail in appendix A. We then generalize the result to arbitrary NMHV amplitudes using a rather short argument based on SUSY Ward identities. This result implies that there is a valid MHV vertex expansion for any NMHV amplitude in $\mathcal{N}=4$ SYM theory.

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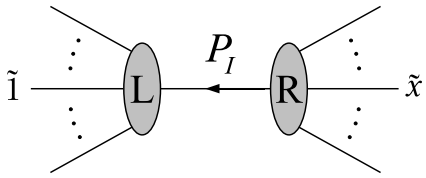

Figure 1. Diagrammatic expansion of an amplitude $A_n(1^-,\ldots,x,\ldots,n)$ under a 2-line shift $[1^-,x\rangle$.

3.1 Valid shifts for $A_n(1^-,\ldots,m_2,\ldots,m_3,\ldots,n)$

We must start with a correct representation of the amplitude $A_n(1^-,\ldots,m_2,\ldots,m_3,\ldots,n)$ which we can use to study the limit $z\to\infty$ under the 3-line shift [8] of the spinors $|1]$, $|m_2]$, $|m_3]$ given by

$$\begin{aligned} |1] \to\ |\hat{1}] &= |1] + z\,\langle m_2 m_3\rangle\,|X]\,, \\ |m_2] \to |\hat{m}_2] &= |m_2] + z\,\langle m_3\,1\rangle\,|X]\,, \\ |m_3] \to |\hat{m}_3] &= |m_3] + z\,\langle 1\,m_2\rangle\,|X]\,, \end{aligned} \tag{3.1}$$

where $|X]$ is an arbitrary reference spinor. Angle bracket spinors $|1\rangle$, $|m_2\rangle$, $|m_3\rangle$ are not shifted. It is assumed that the states m_2 and m_3 share at least one common SU(4) index. We must show that the large z limit of the amplitude deformed by this shift vanishes for all $|X]$. The amplitude then contains no pole at ∞ and Cauchy's theorem can be applied to derive a recursion relation containing a sum of diagrams, each of which is a product of two MHV subdiagrams connected by one internal line. This recursion relation agrees with the MHV vertex expansion of [1].

The representation we need was recently established by Cheung [7] who showed that every amplitude $A_n(1^-,\ldots,x,\ldots,n)$, with particle 1 a negative helicity gluon and others arbitrary, vanishes in the large z limit of the 2-line shift

$$|\tilde{1}] = |1] + z|x]\,, \qquad |\tilde{1}\rangle = |1\rangle\,, \qquad\qquad |\tilde{x}] = |x]\,, \qquad |\tilde{x}\rangle = |x\rangle - z|1\rangle\,. \tag{3.2}$$

This leads to a recursion relation containing a sum of diagrams which are each products of a Left subdiagram, whose n_L lines include the shifted line $\tilde{1}$ and a Right subdiagram whose n_R lines include $\tilde{x}$. Clearly, $n_L + n_R = n + 2$. See figure 1. As explained in appendix A, *only the following two types of diagrams contribute* to the recursion relation:

Type A: MHV $\times$ MHV diagrams with $n_L \geq 3$ and $n_R \geq 4$.

Type B: NMHV $\times$ anti-MHV diagrams with $n_L = n-1$ and $n_R = 3$.

Our strategy is to consider the effect of the shift (3.1) as a secondary shift on each diagram of the recursion relation above. The action of the shift depends on how m_2 and m_3 are placed on the left and right subdiagrams. In appendix A, we show that every Type A diagram vanishes as $z\to\infty$, and that Type B diagrams can be controlled by induction on

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n. Thus the full amplitude $A_n(1^-,\ldots,m_2,\ldots,m_3,\ldots,n)$, with lines m_2 and m_3 sharing a common SU(4) index, is shown to fall off at least as fast as $1/z$ under the $[1,m_2,m_3|$-shift. The full argument is complex and requires detailed examination of special cases for $n=6,7$. Interested readers are referred to appendix A.

We used a 2-line shift simply to have a correct representation of the amplitude to work with in the proof of the large z falloff. That shift plays no further role. In the following we use a more general designation in which line 1 is relabeled m_1.

3.2 Valid shifts for $A_n(m_1,\ldots,m_2,\ldots,m_3,\ldots)$

We now wish to show that any NMHV amplitude vanishes at least as fast as $1/z$ under the 3-line shift

$$|\hat{m}_1] = |m_1] + z\langle m_2 m_3\rangle |X]\,, \qquad |\hat{m}_2] = |m_2] + z\langle m_3 m_1\rangle |X]\,, \qquad |\hat{m}_3] = |m_3] + z\langle m_1 m_2\rangle |X]\,, \tag{3.3}$$

provided that the 3 lines m_1, m_2, m_3 have at least one common SU(4) index which we denote by a. In the previous section we showed that the shift is valid if $r_1 = 4$, where, as usual, r_1 denotes the η-count of line m_1.

We work with SUSY Ward identities and proceed by (finite, downward) induction on r_1. We assume that $1/z$ falloff holds for all amplitudes with $r_1 = \bar{r}$, for some $1 \le \bar{r} \le 4$. We now want to show that it also holds for amplitudes with $r_1 = \bar{r} - 1$. Since $r_1 < 4$, there is at least one SU(4) index not carried by the annihilation operator $\mathcal{O}(m_1)$. We denote this index by b and use $\mathcal{O}^b(m_1)$ to denote the operator of rank $\bar{r}$ containing the original indices of $\mathcal{O}(m_1)$ plus b. This operator satisfies $[\tilde{Q}_b, \mathcal{O}^b(m_1)] = \langle \epsilon\, m_1\rangle \mathcal{O}(m_1)$ (no sum on b). The Ward identity we need (with $|\epsilon\rangle$ chosen such that $\langle \epsilon\, m_1\rangle \neq 0$) is

$$\begin{aligned} 0 &= \langle [\tilde{Q}_b\,,\, \mathcal{O}^b(m_1)\ldots\mathcal{O}(m_2)\ldots\mathcal{O}(m_3)\ldots]\rangle \\ &= \langle \epsilon\, m_1\rangle \langle \mathcal{O}(m_1)\ldots\mathcal{O}(m_2)\ldots\mathcal{O}(m_3)\ldots\rangle + \langle \mathcal{O}^b(m_1)[\tilde{Q}_b,\ldots\mathcal{O}(m_2)\ldots\mathcal{O}(m_3)\ldots]\rangle\,. \end{aligned} \tag{3.4}$$

The first term in the final equality contains the NMHV amplitude we are interested in (which is an amplitude with $r_1 = \bar{r} - 1$). The index b appears 3 times among the operators $\mathcal{O}(i)$ in the commutator in the second term, so there are 3 potentially non-vanishing terms in that commutator. Each term contains an unshifted angle bracket $\langle \epsilon\, i\rangle$ times an NMHV amplitude with $r_1 = \bar{r}$ and lines m_1, m_2, m_3 sharing the common index a, thus it is an amplitude which vanishes as $1/z$ or faster. We conclude

$$\begin{aligned} A_n = \big\langle \mathcal{O}(m_1)\ldots\mathcal{O}(m_2)\ldots\mathcal{O}(m_3)\ldots\big\rangle \to \frac{1}{z} \quad &\text{under the 3-line shift (3.3)} \\ \text{if lines } m_1,\, m_2,\, m_3 &\text{ share at least one SU(4) index}\,. \end{aligned} \tag{3.5}$$

We have thus established valid $[m_1,m_2,m_3|$-shifts if the common index criterion is satisfied. One may ask if this is a necessary condition. There are examples of shifts not satisfying our criterion but which still produce $1/z$ falloff. In [11] the falloff of the 6-gluon amplitude $A_6(1^-,2^-,3^-,4^+,5^+,6^+)$ under 3-line shifts was studied numerically. The results in (6.49) of [11] show that some shifts of three lines which do not share a common index do nonetheless give $1/z$ falloff while others are $O(1)$ at large z.

Note that the 6-gluon amplitude above has a unique shift satisfying our criterion, while any 6-point NMHV amplitude in which the 12 indices appear on 4 or more lines has several such shifts. The case $A_6 = \langle A^{1234}(1)A^{1234}(2)A^{123}(3)A^4(4)A(5)A(6)\rangle$ is one of many examples. Both $[1,2,3|$ and $[1,2,4|$ are valid shifts in this case.

4 Generating functions

A generating function for MHV tree amplitudes in $\mathcal{N}=4$ SYM theory was invented by Nair [9]. The construction was extended to the NMHV level by Georgiu, Glover, and Khoze [10]. Generating functions are a very convenient way to encode how an amplitude depends on the helicity and global symmetry charges of the external states. The generating function for an n-point amplitude depends on $4n$ real Grassmann variables η_{ia}, and the spinors $|i\rangle$, $|i]$ and momenta p_i of the external lines. A 1:1 correspondence between states of the theory and Grassmann derivatives was defined in [11] and given above in (2.8). Any desired amplitude is obtained by applying the product of differential operators associated with its external particles to the generating function. It was also shown in [11] that amplitudes obtained from the generating function obey SUSY Ward identities.

The discussion below is in part a review, but we emphasize the shift-independent universal property of the NMHV generating function. We follow [11], and more information can be found in that reference.

4.1 MHV generating function

The MHV generating function is[4]

$$F_n = \left(\prod_{i=1}^{n}\langle i,i+1\rangle\right)^{-1} \delta^{(8)}\left(\sum_{i=1}^{n}|i\rangle\eta_{ia}\right). \tag{4.1}$$

The 8-dimensional δ-function can be expressed as the product of its arguments, i.e.

$$\delta^{(8)}\left(\sum_{i=1}^{n}|i\rangle\eta_{ia}\right) = \frac{1}{16}\prod_{a=1}^{4}\sum_{i,j=1}^{n}\langle i\,j\rangle\,\eta_{ia}\,\eta_{ja}\,. \tag{4.2}$$

In section 2 we saw that MHV amplitudes $\langle\mathcal{O}(1)\mathcal{O}(2)\ldots\mathcal{O}(n)\rangle$ contain products of operators with a total of 8 SU(4) indices (with each index value appearing exactly twice among the $\mathcal{O}(i)$). The associated product of differential operators D_i from (2.8) has total order 8 and the amplitude may be expressed as:

$$\langle\mathcal{O}(1)\mathcal{O}(2)\ldots\mathcal{O}(n)\rangle = D_1D_2\ldots D_n\,F_n \tag{4.3}$$

$$= \frac{\langle\,\rangle\langle\,\rangle\langle\,\rangle\langle\,\rangle}{\langle12\rangle\langle23\rangle\ldots\langle n-1,n\rangle\langle n1\rangle} \tag{4.4}$$

The numerator is the spin factor which is the product of 4 angle brackets from the differentiation of $\delta^{(8)}$. It is easy [11] to compute spin factors. Here is an example of a 5-point function:

$$\langle A^{1234}(1)A^{1}(2)A^{23}(3)A(4)A^{4}(5)\rangle = \frac{\langle12\rangle\langle13\rangle^2\langle15\rangle}{\langle12\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle51\rangle} \tag{4.5}$$

[4] Lines are identified periodically, $i \equiv i+n$.

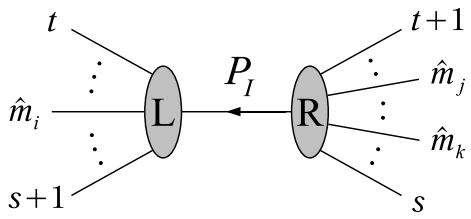

Figure 2. A generic MHV vertex diagram of an NMHV amplitude $A_n(m_1,\dots,m_2,\dots,m_3,\dots)$, arising from a 3-line shift $[m_1,m_2,m_3|$. The set of lines $\hat{m}_i$, $\hat{m}_j$, $\hat{m}_k$ is a cyclic permutation of $\hat{m}_1$, $\hat{m}_2$, $\hat{m}_3$.

Like brackets are not cancelled because we want to illustrate how this example conforms to the general structure above.

4.2 The MHV vertex expansion of an NMHV amplitude

The NMHV generating function is closely tied to the MHV vertex expansion of [1]. The diagrams of such an expansion contain products of two MHV subamplitudes with at least one shifted line in each factor. For n-gluon NMHV amplitudes it was shown in [8] that this expansion agrees with the recursion relation obtained from the 3-line shift (3.3). For a general NMHV amplitude the recursion relation from any valid shift also leads to an expansion containing diagrams with two shifted MHV subamplitudes. In $\mathcal{N}=4$ SYM theory this expansion has the following important property which we demonstrate below; *the recursion relation obtained from any valid 3-line shift contains no reference to the shift used to derive it.* Therefore, all $[m_1,m_2,m_3|$-shifts in which the shifted lines contain at least one common SU(4) index yield the same recursion relation! The MHV vertex expansion is thus unique for every amplitude.

A typical MHV vertex diagram is illustrated in figure 2. In our conventions all particle lines are regarded as outgoing. Therefore, if the particle on the internal line carries a particular set of SU(4) indices of rank r_I in the left subamplitude, it must carry the complementary set of indices of rank $4-r_I$ in the right amplitude. Since each subamplitude must be SU(4) invariant, there is a unique state of the theory which can propagate across the diagram. Any common index a of the shifted lines m_1,m_2,m_3, must also appear on the internal particle in the subdiagram that contains only one shifted line.

Let us assume, as indicated in figure 2, that the left subamplitude contains the external lines $s+1,\dots,t$, including a shifted line $\hat{m}_i$, and that the right subamplitude contains lines $t+1,\dots,s$ and the remaining shifted lines $\hat{m}_j,\hat{m}_k$ (here i,j,k denotes a cyclic permutation of $1,2,3$). In each subamplitude one uses the CSW prescription for the angle spinor of the internal line:

$$|P_I\rangle \equiv P_I|X] = \sum_{i=s+1}^{t} |i\rangle[i\,X]\,, \qquad |-P_I\rangle = -|P_I\rangle\,. \tag{4.6}$$

The contribution of the diagram to the expansion is simply the product of the MHV

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subamplitudes times the propagator of the internal line. It is given by:

$$\frac{(\langle\ \rangle\langle\ \rangle\langle\ \rangle\langle\ \rangle)_L}{\langle -P_I, s+1\rangle\cdots\langle t-1, t\rangle\langle t, -P_I\rangle}\,\frac{1}{P_I^2}\,\frac{(\langle\ \rangle\langle\ \rangle\langle\ \rangle\langle\ \rangle)_R}{\langle P_I, t+1\rangle\cdots\langle s-1, s\rangle\langle s, P_I\rangle} \tag{4.7}$$

The numerator factors are products of 4 angle brackets which are the spin factors for the left and right subamplitudes. They depend on the spinors $|i\rangle$ and $|\pm P_I\rangle$ in each subamplitude and can be calculated easily from the MHV generating function described in section 4.1. The denominators contain the same cyclic products of $\langle i, i+1\rangle$ well known from the Parke-Taylor formula [14], and the standard propagator factor $P_I^2 = (p_{s+1} + \cdots + p_t)^2$.

The main point is that there is simply no trace of the initial shift in the entire formula (4.7) because

i. only angle brackets are involved, and they are unshifted, and

ii. the propagator factor is unshifted.

To complete the discussion we suppose that there is a another valid shift on lines $[m_1', m_2', m_3'|$ which have a common index we will call b. Consider any diagram that appears in the expansion arising from the original $[m_1, m_2, m_3|$-shift. If each subdiagram happens to contain (at least) one of the m_i' lines, then the same diagram with the same contribution to the amplitude occurs in the expansion obtained from the m_i' shift. A diagram from the m_i expansion in which all 3 m_i' lines are located in one of the two subamplitudes cannot occur because the index b would appear 3 times in that subamplitude. This is impossible because that subamplitude is MHV and contains each SU(4) index only twice. This completes the argument that any valid 3-line shift yields the same MHV vertex expansion in which the contribution of each diagram is independent of the chosen shift. The MHV vertex expansion of any NMHV amplitude is unique.

The contribution of each diagram to the expansion depends on the reference spinor $|X]$. Since the physical amplitude contains no such arbitrary object, the sum of all diagrams must be independent of $|X]$. This important fact is guaranteed by the derivation of the recursion relation provided that the amplitude vanishes as $z \to \infty$ for all $|X]$. This is what we proved in section 3.

4.3 The universal NMHV generating function

To obtain the generating function for the (typical) MHV vertex diagram in figure 2 we start with the product of MHV generating functions for each sub-diagram times the internal propagator. We rewrite this product as

$$\frac{1}{\prod_{i=1}^{n}\langle i, i+1\rangle}\, W_I\, \delta^{(8)}(L)\delta^{(8)}(R) \tag{4.8}$$

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with

$$W_I = \frac{\langle s, s+1\rangle\langle t, t+1\rangle}{\langle s\, P_I\rangle\langle s+1, P_I\rangle\, P_I^2\, \langle t\, P_I\rangle\langle t+1, P_I\rangle} \tag{4.9}$$

$$L = |-P_I\rangle\, \eta_{Ia} + \sum_{i=s+1}^{t} |i\rangle\, \eta_{ia} \tag{4.10}$$

$$R = |P_I\rangle\, \eta_{Ia} + \sum_{j=t+1}^{s} |j\rangle\, \eta_{ja}\,. \tag{4.11}$$

The Grassmann variable η_{Ia} is used for the internal line. We have separated the denominator factors in (4.7) into a Parke-Taylor cyclic product over the full set of external lines times a factor W_I involving the left-right split, as used in [10].

The contribution of (4.8) to the diagram for a given process is then obtained by applying the appropriate product of Grassmann derivatives from (2.8). This product includes derivatives for external lines and the derivatives $D_{I_L} D_{I_R}$ for the internal lines. It follows from the discussion above that the operators D_{I_L} and D_{I_R} are of order r_I and $4 - r_I$ respectively, and that their product is simply

$$D_{I_L} D_{I_R} = \prod_{a=1}^{4} \frac{\partial}{\partial \eta_{Ia}}\,. \tag{4.12}$$

We apply this 4th order derivative to (4.8), convert the derivative to a Grassmann integral as in [11], and integrate using the formula [10]

$$\int \prod_{a=1}^{4} d\eta_{Ia} \delta^{(8)}(L) \delta^{(8)}(R) = \delta^{(8)}\left(\sum_{i=1}^{n} |i\rangle \eta_{ia}\right) \prod_{b=1}^{4} \sum_{j=s+1}^{t} \langle P_I j\rangle \eta_{jb}\,. \tag{4.13}$$

Thus we obtain the generating function

$$\begin{aligned} \mathcal{F}_{I,n} &= \frac{\delta^{(8)}\left(\sum_{i=1}^{n} |i\rangle\eta_{ia}\right)}{\prod_{i=1}^{n}\langle i, i+1\rangle}\, W_I \prod_{b=1}^{4} \sum_{i=s+1}^{t} \langle P_I\, i\rangle \eta_{ib} \\ &= \frac{\delta^{(8)}\left(\sum_{i=1}^{n} |i\rangle\eta_{ia}\right)}{\prod_{i=1}^{n}\langle i, i+1\rangle}\, W_I \prod_{b=1}^{4} \sum_{j=t+1}^{s} \langle P_I\, j\rangle \eta_{jb} \end{aligned} \tag{4.14}$$

The two expressions are equal because $\delta^{(8)}$ for the external lines is present. Using (4.2) one can see that (4.14) contains a sum of terms, each containing a product of 12 η_{ia}. To obtain the contribution of the diagram to a particular NMHV process we simply apply the appropriate product of differential operators of total order 12. This gives the value of the diagram in the original form (4.7).

In section 4.2 we argued that the MHV vertex expansion of any particular amplitude is unique and contains exactly the diagrams which come from the recursion relation associated with any valid 3-line shift $[m_1, m_2, m_3|$ which satisfies the common index criterion. A diagram is identified by specifying the channel in which a pole occurs. A 6-point amplitude

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$A_6(1,2,3,4,5,6)$ can contain 2-particle poles in the channels $(12),(23),(34),(45),(56)$, or (61), and there can be 3-particle poles in the channels $(123),(234),(345)$. However, different 6-point NMHV amplitudes contain different subsets of the 9 possible diagrams. For example, the 6-gluon amplitudes with helicity configurations $A_6(---+++)$ and $A_6(-+-+-+)$ each have one valid common index shift of the 3 negative helicity lines. In the first case, there are 6 diagrams, since diagrams with poles in the (45), (56) and (123) channels do not occur in the recursion relation, but all 9 possible diagrams contribute to the recursion relation for the second case. The amplitude $\langle A^1(1)A^{12}(2)A^{23}(3)A^{234}(4)A^{134}(5)A^4(6)\rangle$ for a process with 4 gluinos and 2 scalars is a more curious example; its MHV vertex expansion contains only one diagram with pole in the (45) channel.

We would like to define a universal generating function which contains the amplitudes for *all* n-point NMHV amplitudes, such that any particular amplitude is obtained by applying the appropriate 12th order differential operator. It is natural to define the generating function as

$$\mathcal{F}_n = \sum_I \mathcal{F}_{I,n} \tag{4.15}$$

in which we sum the generating functions (4.14) for *all* $n(n-3)/2$ *possible* diagrams that can appear in the MHV vertex expansion of n-point amplitudes, for example all 9 diagrams listed above for 6-point amplitudes. If a particular diagram $\bar{I}$ does not appear in the MHV vertex expansion of a given amplitude, then the spin factor obtained by applying the appropriate Grassmann differential operator to the generating function $\mathcal{F}_{\bar{I},n}$ must vanish, leaving only the actual diagrams which contribute to the expansion.

To convince the reader that this is true, we first make an observation which follows from the way in which each $\mathcal{F}_{I,n}$ is constructed starting from (4.8). We observe that the result of the application of a Grassmann derivative $\mathcal{D}^{(12)}$ of order 12 in the external η_{ia} to any $\mathcal{F}_{I,n}$ is the same as the result of applying the operator $\mathcal{D}^{(16)} = \mathcal{D}^{(12)} D_{I_L} D_{I_R}$ to the product in (4.8). If non-vanishing, this result is simply the product of the spin factors for the left and right subdiagrams, so the contribution of the diagram I to the amplitude corresponding to $\mathcal{D}^{(12)}$ is correctly obtained.

We now show that $\mathcal{D}^{(12)}\mathcal{F}_{\bar{I},n}$ vanishes when a diagram $\bar{I}$ does not contribute to the corresponding amplitude. We first note that the amplitudes governed by $\mathcal{F}_n$ are all NMHV. Thus they all have overall η-count 12, and SU(4) invariance requires that each index value $a = 1,2,3,4$ must appear exactly 3 times among the external lines. Denote the lines which carry the index a in the amplitude under study by q_{1a}, q_{2a}, q_{3a}. Consider a diagram I and suppose that for every value of a its two subamplitudes each contain at least one line from the set $q_{ka},\ k = 1,2,3$. Then the diagram I appears in the MHV vertex expansion of the amplitude, and the diagram contributes correctly to $\mathcal{D}^{(12)}\mathcal{F}_n$. The other possibility is that there is a diagram $\bar{I}$ such that for some index value b the 3 lines q_{1b}, q_{2b}, q_{3b} appear in only one subamplitude, say the left subamplitude. Then the right subamplitude cannot be SU(4) invariant. Its spin factor vanishes and the diagram does not contribute.

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5 Spin state sums for loop amplitudes

Consider the L-loop amplitude shown in figure 3. The evaluation of the $(L+1)$-line unitarity cut involves a sum over all intermediate states that run in the loops. The generating functions allow us to do such sums very efficiently, for any arrangements of external states as long as the left and right subamplitudes, denoted I and J, are either MHV or NMHV tree amplitudes.

We begin by a general analysis of cut amplitudes of the type in figure 3. Assume that the full amplitude is $\mathrm{N}^k\mathrm{MHV}$. Then the total η-count is $\sum_{\text{ext}\,i} r_i = 4(k+2)$. Let the η-count of the lth loop state on the subamplitude I be w_l; then that same line will have η-count $4 - w_l$ on the subamplitude J. The total η-counts on the subamplitudes I and J are then, respectively,

$$r_I = \sum_{\text{ext}\,i\in I} r_i + \sum_{l=1}^{L+1} w_l\,, \qquad r_J = \sum_{\text{ext}\,j\in J} r_j + \sum_{l=1}^{L+1} (4 - w_l)\,, \tag{5.1}$$

so that

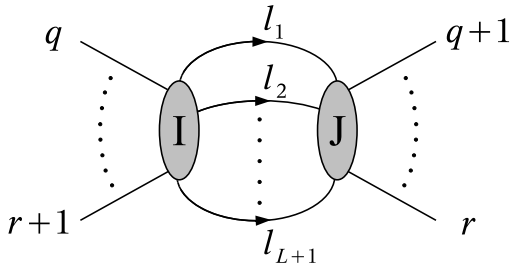
$$r_I + r_J = \sum_{\text{ext}\,i} r_i + 4(L+1) = 4(k+L+3)\,. \tag{5.2}$$

Each subamplitude I and J must have an η-count $r_{I,J}$ which is a multiple of 4. If the overall amplitude is MHV and $L = 1$, then (5.2) gives $r_I + r_J = 16$, and the only possibility is that both subamplitudes I and J are MHV with η-counts 8 each. (Total η-count 4 is non-vanishing only for a 3-point anti-MHV amplitude; such spin sums are considered in section 6.) Likewise, a 2-loop MHV amplitude has $r_I + r_J = 20 = 8 + 12 = 12 + 8$, so the intermediate state sum splits into MHV $\times$ NMHV plus NMHV $\times$ MHV.

The table in figure 3 summarizes the possibilities for MHV and NMHV loop amplitudes with $(L+1)$-line cuts. For each split, one must sum over all intermediate states; the tree generating functions allow us to derive new generating functions for cut amplitudes with all intermediate states summed.

We outline the general strategy before presenting the detailed examples. Let $\mathcal{F}_I$ and $\mathcal{F}_J$ be generating functions for the subamplitudes I and J of the cut amplitude. To evaluate the cut, we must act on the product $\mathcal{F}_I\,\mathcal{F}_J$ with the differential operators of all the external states $D_{\text{ext}}^{(4k+8)}$ and of all the internal states $D_1 D_2 \cdots D_{L+1}$. The fourth order differential operators of the internal lines distribute themselves in all possible ways between $\mathcal{F}_I$ and $\mathcal{F}_J$ and thus automatically carry out the spin sum. In [11] it was shown how to evaluate the 1-loop MHV state sums when the external lines where all gluons. This was done by first acting with the derivative operators of the external lines, and then evaluating the derivatives for the loop states. We generalize the approach here to allow any set of external states of the $\mathcal{N} = 4$ theory. This is done by postponing the evaluation of the external state derivatives, and instead carrying out the the internal line Grassmann derivatives by converting them to Grassmann integrations. The result is a generating function $\mathcal{F}_{\text{cut}}$ for the cut amplitude. It is defined as

$$\mathcal{F}_{\text{cut}} \;=\; D_1 D_2 \cdots D_{L+1}\;\mathcal{F}_I\,\mathcal{F}_J\,. \tag{5.3}$$



external		$L=1$	$L=2$	$L=3$	$L=4$
MHV	$\to$	MHV $\times$ MHV	MHV $\times$ NMHV NMHV $\times$ MHV	MHV $\times$ N^{2}MHV NMHV $\times$ NMHV N^{2}MHV $\times$ MHV	MHV $\times$ N^{3}MHV NMHV $\times$ N^{2}MHV N^{2}MHV $\times$ NMHV N^{3}MHV $\times$ MHV
NMHV	$\to$	MHV $\times$ NMHV NMHV $\times$ MHV	MHV $\times$ N^{2}MHV NMHV $\times$ NMHV N^{2}MHV $\times$ MHV	etc	

Figure 3. N^{k}MHV loop amplitude evaluated by a unitarity cut of $(L+1)$-lines. The sum over intermediate states involves all subamplitudes I and J with η-counts r_I and r_J such that $r_I + r_J = 4(k+L+3)$. (For $L=1$ we assume that I and J each have more than one external leg, so that 3-point anti-MHV does not occur in the spin sum.)

The value of a particular cut amplitude is found by applying the external state differential operators $D_{\rm ext}^{(4k+8)}$ to $\mathcal{F}_{\rm cut}$.

In the following we derive generating functions for unitarity cuts of 1-, 2- and 3-loop MHV and NMHV amplitudes. Spin sums involving N^{2}MHV and N^{3}MHV subamplitudes for $L = 3, 4$ are carried out using anti-MHV and anti-NMHV generating functions in section 6.

5.1 1-loop intermediate state sums

5.1.1 1-loop MHV $\times$ MHV

Consider the intermediate state sum in a 2-line cut 1-loop amplitude. Let the external states be any $\mathcal{N}=4$ states such that the full loop amplitude is MHV. By the analysis above, the subamplitudes I and J of the cut loop amplitude must then also be MHV.

We first calculate the intermediate spin sum and then include the appropriate prefactors. The state dependence of an MHV subamplitude is encoded in the $\delta^{(8)}$-factor of the MHV generating function. We will refer to the sum over spin factors as the "spin sum factor" of the cut amplitude. For the present case, the spin sum factor is

$$D_1 D_2\, \delta^{(8)}(I)\, \delta^{(8)}(J) \tag{5.4}$$

with $D_i = \prod_{a=1}^4 \partial/\partial\eta_{ia}$ being the 4th order derivatives associated with the internal line l_i

and

$$I = |l_1\rangle\eta_{1a} + |l_2\rangle\eta_{2a} + \sum_{\text{ext } i\in I} |i\rangle\eta_{ia}\,,$$
$$J = -|l_1\rangle\eta_{1a} - |l_2\rangle\eta_{2a} + \sum_{\text{ext } j\in J} |j\rangle\eta_{ja}\,. \tag{5.5}$$

We proceed by converting the D_1D_2 Grassmann differentiations to integrations. Perform first the integration over η_2 to find [10]

$$\begin{aligned} D_1D_2\,\delta^{(8)}(I)\,\delta^{(8)}(J) &= \int d^4\eta_1\,d^4\eta_2\,\delta^{(8)}(I)\,\delta^{(8)}(J) \\ &= \delta^{(8)}(I+J)\int d^4\eta_1 \prod_{a=1}^{4}\left(\sum_{\text{ext } j\in J}\langle l_2\,j\rangle\,\eta_{ja} - \langle l_2\,l_1\rangle\,\eta_{1a}\right). \end{aligned} \tag{5.6}$$

The delta-function $\delta^{(8)}(I+J)$ involves only the sum over external states and does therefore not depend on η_1. The η_1-integrations picks up the η_1 term only, so we simply get

$$D_1D_2\,\delta^{(8)}(I)\,\delta^{(8)}(J) = \langle l_1\,l_2\rangle^4\,\delta^{(8)}\left(\sum_{\text{all ext } m}|m\rangle\eta_{ma}\right). \tag{5.7}$$

If the external states are two negative helicity gluons i and j and the rest are positive helicity gluons, then, no matter where the gluons i and j are placed, we get $\langle l_1\,l_2\rangle^4\langle i\,j\rangle^4$, in agreement with (5.6) of [15] and (4.9) of [11].

Let us now include also the appropriate pre-factors in the generating function. For the MHV subamplitudes these are simply the cyclic products of momentum angle brackets. Collecting the cyclic product of *external* momenta, we can write the full generating function for the MHV × MHV 1-loop generating function as

$$\mathcal{F}^{\text{1-loop}}_{\text{MHV}\times\text{MHV}} = \frac{\langle q,q+1\rangle\langle r,r+1\rangle\langle l_1l_2\rangle^2}{\langle q\,l_1\rangle\langle q+1,l_1\rangle\langle r\,l_2\rangle\langle r+1,l_2\rangle}\,\frac{1}{\prod_{\text{ext }i}\langle i,i+1\rangle}\,\delta^{(8)}\left(\sum_{\text{all ext } m}|m\rangle\eta_{ma}\right),$$

or simply,

$$\mathcal{F}^{\text{1-loop}}_{\text{MHV}\times\text{MHV}} = \frac{\langle q,q+1\rangle\langle r,r+1\rangle\langle l_1l_2\rangle^2}{\langle q\,l_1\rangle\langle q+1,l_1\rangle\langle r\,l_2\rangle\langle r+1,l_2\rangle}\,\mathcal{F}^{\text{tree}}_{\text{MHV}}(\text{ext})\,. \tag{5.8}$$

Note that the state dependence of the cut MHV × MHV amplitude is included entirely in the MHV generating function, and all dependence on the loop momentum is in the prefactor.

5.1.2 Triple cut of NMHV 1-loop amplitude: MHV × MHV × MHV

In this section we evaluate the intermediate state sum for a 1-loop NMHV amplitude with a triple cut, as illustrated in figure 4. The triple cut is different from the cuts considered at

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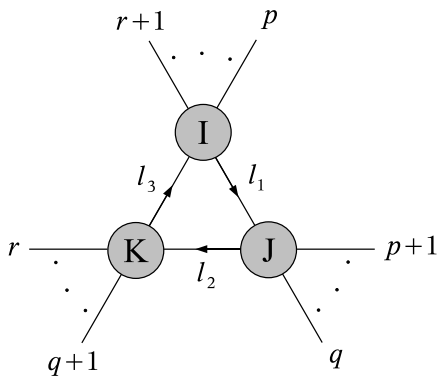

Figure 4. Triple cut of NMHV 1-loop amplitude gives MHV subamplitudes I, J, and K.

the beginning of section 5. Its primary feature is that it gives three subamplitudes which are all MHV. To see this, note that the η-counts of the subamplitudes I, J and K are

$$r_I = \sum_{\text{ext } i\in I} r_i + w_1 + 4 - w_3\,, \quad r_J = \sum_{\text{ext } j\in J} r_j + w_2 + 4 - w_1\,, \quad r_K = \sum_{\text{ext } k\in K} r_k + w_3 + 4 - w_2\,, \tag{5.9}$$

where the r_i are the η-counts of the external states and w_l and $4 - w_l$ are the η-counts at each end of the internal lines. Since the full amplitude is NMHV, we have

$$r_I + r_J + r_K = \sum_{\text{all ext } i} r_i + 12 = 24\,. \tag{5.10}$$

We now assume that each subamplitude I, J, and K has more than three legs and thus more than one external leg. Then (5.10) has only one solution, namely $r_I = r_J = r_K = 8$, so each subamplitude is MHV.

Let us again first evaluate the spin sum and include the appropriate prefactors at the end. The spin sum factor is calculated by letting the differential operators of the internal states act on the product of the three MHV generating functions for the subamplitudes. We have

$$f^{\text{triple}} = D_1 D_2 D_3 \big[\delta^{(8)}(I)\ \delta^{(8)}(J)\ \delta^{(8)}(K)\big]\,, \tag{5.11}$$

where

$$\begin{aligned} I &= |l_1\rangle\eta_{1a} - |l_3\rangle\eta_{3a} + \sum_{\text{ext } i\in I} |i\rangle\eta_{ia}\,, \\ J &= -|l_1\rangle\eta_{1a} + |l_2\rangle\eta_{2a} + \sum_{\text{ext } j\in J} |j\rangle\eta_{ja}\,, \\ K &= -|l_2\rangle\eta_{2a} + |l_3\rangle\eta_{3a} + \sum_{\text{ext } k\in K} |k\rangle\eta_{ka}\,. \end{aligned} \tag{5.12}$$

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Again we convert the differentiations to integrations, and perform the integrations one at a time to find

$$
\begin{aligned}
f^{\text{triple}} &= \int d^4\eta_1 d^4\eta_2 d^4\eta_3 \delta^{(8)}(I)\delta^{(8)}(J)\delta^{(8)}(K) \\
&= \int d^4\eta_2 d^4\eta_3 \delta^{(8)}(I+J)\delta^{(8)}(K) \prod_{a=1}^{4} \left(\sum_{\text{ext } i\in I} \langle l_1 i\rangle \eta_{ia} - \langle l_1 l_3\rangle \eta_{3a} \right) \\
&= \delta^{(8)}(I+J+K) \prod_{a=1}^{4} \left(\sum_{\text{ext } i\in I} \langle l_1 i\rangle \eta_{ia} - \langle l_1 l_3\rangle \eta_{3a} \right) \left(\sum_{\text{ext } k\in K} \langle l_2 k\rangle \eta_{ka} + \langle l_2 l_3\rangle \eta_{3a} \right) \\
&= \delta^{(8)} \left(\sum_{\text{all ext m}} |m\rangle \eta_{mb} \right) \prod_{a=1}^{4} \left(- \sum_{\text{ext } k\in K} \langle l_3 l_1\rangle \langle l_2 k\rangle \eta_{ka} + \sum_{\text{ext } i\in I} \langle l_2 l_3\rangle \langle l_1 i\rangle \eta_{ia} \right). \qquad (5.13)
\end{aligned}
$$

This is the generating function for the spin sum factor of the triple cut.[5]

If the external particles are all gluons with three negative helicity gluons i', j', k' distributed on the cut with $i' \in I$, $j' \in J$, and $k' \in K$, then the triple cut spin sum factor is

$$
\begin{aligned}
& D_{i'} D_{j'} D_{k'} f^{\text{triple}} \\
&\quad = D_{i'} D_{k'} \left\{ \left[D_{j'} \delta^{(8)} \left(\sum_{\text{all ext m}} |m\rangle \eta_{mb} \right) \right] \prod_{a=1}^{4} \left(- \langle l_3 l_1\rangle \langle l_2 k'\rangle \eta_{k'a} + \langle l_2 l_3\rangle \langle l_1 i'\rangle \eta_{i'a} + \dots \right) \right\} \\
&\quad = D_{i'} D_{k'} \prod_{a=1}^{4} \left(\langle j' i'\rangle \eta_{i'a} + \langle j' k'\rangle \eta_{k'a} + \dots \right) \left(- \langle l_3 l_1\rangle \langle l_2 k'\rangle \eta_{k'a} + \langle l_2 l_3\rangle \langle l_1 i'\rangle \eta_{i'a} + \dots \right) \\
&\quad = \left(\langle j' i'\rangle \langle l_1 l_3\rangle \langle l_2 k'\rangle - \langle j' k'\rangle \langle l_2 l_3\rangle \langle l_1 i'\rangle \right)^4 \\
&\quad = \left(\langle l_1 j'\rangle \langle l_3 i'\rangle \langle k' l_2\rangle - \langle l_2 j'\rangle \langle l_3 k'\rangle \langle l_1 i'\rangle \right)^4 . \qquad (5.15)
\end{aligned}
$$

This agrees[6] with (4.23) of [16] and (4.13) of [11].

To complete the calculation, we must include the appropriate prefactors. The full MHV × MHV × MHV triple cut 1-loop generating function is then

$$
\begin{aligned}
\mathcal{F}^{\text{1-loop NMHV triple cut}}_{\text{MHV}^3} &= \frac{\langle p, p+1\rangle \langle q, q+1\rangle \langle r, r+1\rangle}{\langle p\, l_1\rangle \langle l_1, p+1\rangle \langle q\, l_2\rangle \langle l_2, q+1\rangle \langle r\, l_3\rangle \langle l_3, r+1\rangle} \times \mathcal{F}^{\text{tree}}_{\text{MHV}}(\text{ext}) \\
&\quad \times \prod_{a=1}^{4} \left(- \sum_{\text{ext } k\in K} \langle l_3 l_1\rangle \langle l_2 k\rangle \eta_{ka} + \sum_{\text{ext } i\in I} \langle l_2 l_3\rangle \langle l_1 i\rangle \eta_{ia} \right). \qquad (5.16)
\end{aligned}
$$

It is interesting to note that the structure of $\mathcal{F}^{\text{tree}}_{\text{MHV}} \times \prod_a \sum$ is very similar to the NMHV generating function for an MHV vertex diagram.

[5] Note that using the overall $\delta^{(8)}$ and the Schouten identity, the $\prod \sum$-factor can be rearranged cyclically as

$$
\begin{aligned}
- \sum_{\text{ext } k\in K} \langle l_3 l_1\rangle \langle l_2 k\rangle \eta_{ka} + \sum_{\text{ext } i\in I} \langle l_2 l_3\rangle \langle l_1 i\rangle \eta_{ia} &= - \sum_{\text{ext } i\in I} \langle l_1 l_2\rangle \langle l_3 i\rangle \eta_{ia} + \sum_{\text{ext } j\in J} \langle l_3 l_1\rangle \langle l_2 j\rangle \eta_{ja} \\
&= - \sum_{\text{ext } j\in J} \langle l_2 l_3\rangle \langle l_1 j\rangle \eta_{ja} + \sum_{\text{ext } k\in K} \langle l_1 l_2\rangle \langle l_3 k\rangle \eta_{ka} . \qquad (5.14)
\end{aligned}
$$

[6] The power in [11, 16] was 8, not 4, because the calculations were done in $\mathcal{N} = 8$ SG instead of $\mathcal{N} = 4$ SYM.

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5.2 MHV 2-loop state sum with NMHV × MHV

As a first illustration of the application of the NMHV generating function, we calculate the intermediate state sum of a 3-line cut 2-loop MHV amplitude. The state sum splits into two separate cases NMHV × MHV and MHV × NMHV (see section 5). It suffices to derive an expression for the generating function of the NMHV × MHV state sum; from that the MHV × NMHV sum is easily obtained by relabeling momenta.

We express the NMHV subamplitude I in terms of its MHV vertex expansion. We denote by I_L each MHV vertex diagram in the expansion, and we also let I_L and I_R label the Left and Right MHV subamplitudes of the diagram. For each MHV vertex diagram $I_L \subset I$ we compute the spin sum factor

$$f_{I_L} = D_1 D_2 D_3 \left[\left(\delta^{(8)}(I) \prod_{a=1}^{4} \sum_{i \in I_L} \langle i P_{I_L} \rangle \eta_{ia} \right) \delta^{(8)}(J) \right]. \tag{5.17}$$

The prefactors of the generating functions will be included later when we sum the contributions of all the diagrams. We are free to define the *left* MHV subamplitude I_L to be the one containing *either one or none* of the loop momenta. For definiteness, let us denote the loop momentum contained in I_L by l_α, the others by l_β, l_γ. (If I_L does not contain any loop momentum, this assignment is arbitrary.) Since $l_\beta, l_\gamma \notin I_L$ we get

$$\begin{aligned}
f_{I_L} &= \int d^4\eta_\alpha d^4\eta_\beta d^4\eta_\gamma \left(\delta^{(8)}(I) \delta^{(8)}(J) \prod_{a=1}^{4} \sum_{i \in I_L} \langle i P_{I_L} \rangle \eta_{ia} \right) \\
&= \delta^{(8)}\big(I + J\big) \int d^4\eta_\alpha \, d^4\eta_\beta \prod_{a=1}^{4} \left(\sum_{i' \in I} \langle i' \gamma \rangle \eta_{i' a} \right) \left(\sum_{i \in I_L} \langle i P_{I_L} \rangle \eta_{ia} \right) \\
&= \delta^{(8)}\big(I + J\big) \int d^4\eta_\alpha d^4\eta_\beta \prod_{a=1}^{4} \Big(\langle \gamma\alpha \rangle \eta_{\alpha a} + \langle \gamma\beta \rangle \eta_{\beta a} + \ldots \Big) \left(\sum_{\text{ext}\, i \in I_L} \langle i P_{I_L} \rangle \eta_{ia} + \delta_{l_\alpha \in I_L} \langle \alpha P_{I_L} \rangle \eta_{\alpha a} \right) \\
&= \delta_{l_\alpha \in I_L} \langle \beta\gamma \rangle^4 \, \langle \alpha P_{I_L} \rangle^4 \, \delta^{(8)} \left(\sum_{\text{all ext}\, k} |k\rangle \eta_{ka} \right).
\end{aligned} \tag{5.18}$$

We have introduced a Kronecker delta $\delta_{l_\alpha \in I_L}$ which is 1 if $l_\alpha \in I_L$ and zero otherwise. If $l_\alpha \notin I_L$, then none of the internal momenta connect to I_L. The calculation shows that such "1-particle reducible" diagrams do not contribute to the spin sum. This is a common feature of all spin sums we have done.

Including now the prefactors and summing over all MHV vertex diagrams $I_L \subset I$, the generating function for the cut 2-loop amplitude is

$$\mathcal{F}^{\text{2-loop, } n\text{-pt}}_{\text{NMHV}\times\text{MHV}} = \frac{1}{\prod_{j \in J} \langle j, j+1 \rangle} \sum_{I_L \subset I} W_{I_L} \frac{1}{\prod_{i \in I} \langle i, i+1 \rangle} f_{I_L} \,, \tag{5.19}$$

where W_{I_L} is the prefactor (4.9). Separating out the dependence on the external states into an overall factor $\mathcal{F}^{\text{tree}}_{\text{MHV}}(\text{ext})$, we get

$$\mathcal{F}^{\text{2-loop, } n\text{-pt}}_{\text{NMHV}\times\text{MHV}} = \mathcal{F}^{\text{tree}}_{\text{MHV}}(\text{ext}) \frac{\langle q, q+1 \rangle \langle r, r+1 \rangle}{\langle q, l_1 \rangle \langle l_1, q+1 \rangle \langle r, l_3 \rangle \langle l_3, r+1 \rangle \langle l_1\, l_2 \rangle^2 \langle l_2\, l_3 \rangle^2} \sum_{I_L \subset I} W_{I_L} \, (S.F.)_{I_L} \tag{5.20}$$

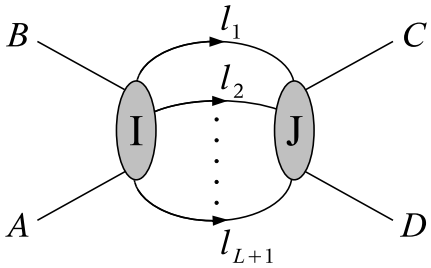

Figure 5. 4-point L-loop MHV amplitude with $(L+1)$-line cut.

with

$$(S.F.)_{I_L} = \langle\beta\gamma\rangle^4 \langle\alpha P_{I_L}\rangle^4 \, \delta_{l_\alpha\in I_L;\; l_\beta, l_\gamma\notin I_L} \,. \tag{5.21}$$

Each term in the sum over $I_L \subset I$ depends on the reference spinor $|X]$ through the prescription $|P_{I_L}\rangle = P_{I_L}|X]$, but the sum of all diagrams must be $|X]$-independent.

Example: 3-line cut of 4-point 2-loop amplitude. Let the external states be A, B, C, D, with A, B on the subamplitude I and C, D on J, as shown in figure 5 with $L = 2$. The subamplitude I of the cut is a 5-point NMHV amplitude. Its MHV vertex expansion has five diagrams $(I_L|I_R)$, which we list with their spin sum factors:

$$\begin{aligned}
&(A, B|l_1, l_2, l_3) && \leftrightarrow \quad 0\,, \\
&(B, l_1|l_2, l_3, A)\,, \quad (A, B, l_1|l_2, l_3) && \leftrightarrow \quad \langle l_2 l_3\rangle^4 \langle l_1 P_{I_L}\rangle^4\,, \\
&(l_3, A|B, l_1, l_2)\,, \quad (l_3, A, B|l_1, l_2) && \leftrightarrow \quad \langle l_1 l_2\rangle^4 \langle l_3 P_{I_L}\rangle^4\,.
\end{aligned} \tag{5.22}$$

We have checked numerically that the sum $\sum_{I_L\subset I} W_{I_L} \, (S.F.)_{I_L}$ is independent of the reference spinor $|X]$.

As a further check, let us assume that the two particles C and D are negative helicity gluons while the two particles A and B are positive helicity gluons. With the assumption that the cut is NMHV $\times$ MHV, there is only one choice for the internal particles: they have to be gluons, negative helicity coming out of the subamplitude I and thus positive helicity on J. So the spin sum only has one term, namely

$$\begin{aligned}
&A_5\big(A^+, B^+, l_1^-, l_2^-, l_3^-\big)\, A_5\big(C^-, D^-, -l_3^+, -l_2^+, -l_1^+\big) \\
&= \frac{[A\,B]^3}{[B\,l_1][l_1\,l_2][l_2 l_3][l_3\,A]} \frac{\langle C\,D\rangle^3}{\langle D\,l_3\rangle\langle l_3\,l_2\rangle\langle l_2\,l_1\rangle\langle l_1\,C\rangle}\,.
\end{aligned} \tag{5.23}$$

This should be compared with the result of the spin sum (5.20) with the appropriate spin state dependence from $\mathcal{F}^{\rm tree}_{\rm MHV}(\text{ext})$. We have checked numerically that the results agree.

We can use this result to replace the spin sum over $I_L \subset I$ in (5.20) by the anti-MHV $\times$ MHV factor (5.23) and then write the full 4-point generating function in the simpler form

$$\mathcal{F}^{\text{2-loop, 4-pt}}_{\rm NMHV\times MHV} = \frac{[A\,B]^3\langle D\,A\rangle\langle A\,B\rangle\langle B\,C\rangle}{\langle C|l_1|B]\langle D|l_3|A]P^2_{l_1 l_2}P^2_{l_2 l_3}} \, \mathcal{F}^{\rm tree}_{\rm MHV}(\text{ext})\,. \tag{5.24}$$

We have checked numerically the agreement between (5.20) and (5.24) for 4 external lines. The generating function (5.24) gives the correct result for any MHV choice of 4 external states.

5.3 MHV 3-loop state sum with NMHV × NMHV

Consider the NMHV × NMHV part of the 3-loop spin sum. We express the I and J subamplitudes in terms of their MHV vertex expansions; thus in the intermediate state sum we must sum over all products of MHV vertex diagrams $I_L \subset I$ and $J_L \subset J$. We first compute the spin sum factor associated with such a product, then include the necessary prefactors in order to get a general expression for the intermediate state sum.

For each MHV vertex diagram of the subamplitudes I and J, there is a freedom in choosing which MHV vertex we call "left". This always allows us to choose I_L and J_L such that neither contains the internal momentum line l_4. This is a convenient choice for performing the η_4 integration first and then evaluating the three other η-integrations. The spin sum factor for a product of MHV vertex diagrams I_L and J_L with $l_4 \notin I_L \cup J_L$ is then

$$\begin{aligned}
&D_1D_2D_3D_4\ \mathcal{F}^{\text{tree-diagram}}_{\text{NMHV}}(I_L)\ \mathcal{F}^{\text{tree-diagram}}_{\text{NMHV}}(J_L)\\
&= D_1D_2D_3 \int d^4\eta_4 \left(\delta^{(8)}(I)\prod_{a=1}^{4}\sum_{i\in I_L}\langle iP_{I_L}\rangle\eta_{ia}\right)\left(\delta^{(8)}(J)\prod_{b=1}^{4}\sum_{j\in J_L}\langle jP_{J_L}\rangle\eta_{jb}\right)\\
&= \delta^{(8)}(I+J)\ D_1D_2D_3\prod_{a=1}^{4}\Big(\langle l_1l_4\rangle\eta_{1a}+\langle l_2l_4\rangle\eta_{2a}+\langle l_3l_4\rangle\eta_{3a}+\dots\Big)\\
&\qquad\qquad \Big(\delta_{l_1\in I_L}\langle l_1P_{I_L}\rangle\eta_{1a}+\delta_{l_2\in I_L}\langle l_2P_{I_L}\rangle\eta_{2a}+\delta_{l_3\in I_L}\langle l_3P_{I_L}\rangle\eta_{3a}+\dots\Big)\\
&\qquad\qquad \Big(\delta_{l_1\in J_L}\langle l_1P_{J_L}\rangle\eta_{1a}+\delta_{l_2\in J_L}\langle l_2P_{J_L}\rangle\eta_{2a}+\delta_{l_3\in J_L}\langle l_3P_{J_L}\rangle\eta_{3a}+\dots\Big)\\
&= \delta^{(8)}(I+J)\ (\text{s.s.f.})_{I_L,J_L}\,,
\end{aligned} \tag{5.25}$$

where

$$(\text{s.s.f.})_{I_L,J_L} = \left[\det\begin{pmatrix} \langle l_1l_4\rangle & \langle l_2l_4\rangle & \langle l_3l_4\rangle \\ \delta_{l_1\in I_L}\langle l_1P_{I_L}\rangle & \delta_{l_2\in I_L}\langle l_2P_{I_L}\rangle & \delta_{l_3\in I_L}\langle l_3P_{I_L}\rangle \\ \delta_{l_1\in J_L}\langle l_1P_{J_L}\rangle & \delta_{l_2\in J_L}\langle l_2P_{J_L}\rangle & \delta_{l_3\in J_L}\langle l_3P_{J_L}\rangle \end{pmatrix}\right]^4 . \tag{5.26}$$

We must sum over all diagrams including the appropriate prefactors. There are W_{I_L} and W_{J_L} factors (4.9) from the two MHV vertex expansions, as well as cyclic products. With momentum labels q, $q+1$ etc as in figure 3 we can write the NMHV × NMHV part of the full 4-line cut 3-loop MHV amplitude as

$$\begin{aligned}
\mathcal{F}^{\text{3-loop},n\text{-pt}}_{\text{NMHV}\times\text{NMHV}} = &-\frac{\langle q,q+1\rangle\langle r,r+1\rangle}{\langle q,l_1\rangle\langle l_1,q+1\rangle\langle r,l_4\rangle\langle l_4,r+1\rangle\langle l_1\,l_2\rangle^2\langle l_2\,l_3\rangle^2\langle l_3\,l_4\rangle^2}\\
&\times\mathcal{F}^{\text{tree}}_{\text{MHV}}(\text{ext})\sum_{I_L\subset I,\ J_L\subset J} W_{I_L}W_{J_L}(\text{s.s.f.})_{I_L,J_L}\,.
\end{aligned} \tag{5.27}$$

The product of any MHV vertex diagrams I_L and J_L involve two independent reference spinors X_I and X_J from the internal momentum prescriptions $|P_{I_L}\rangle = P_{I_L}|X_I]$ and $|P_{J_L}\rangle = P_{J_L}|X_J]$, but the sum over all diagrams must be independent of both reference spinors.

Consider the 4-point 3-loop amplitude. Let the external states be A, B, C, D, with A, B on the subamplitude I and C, D on J, as in figure 5. The NMHV subamplitudes I and J are 6-point functions, so their MHV vertex expansions involve a sum of 9 diagrams. For the subamplitude I these diagrams are listed as $(I_L|I_R)$:

$$\begin{aligned}
&(AB\,|\,l_1l_2l_3l_4)\,, \quad (Bl_1\,|\,l_2l_3l_4A)\,, \quad (l_1l_2\,|\,l_3l_4AB)\,,\\
&(l_2l_3\,|\,l_4ABl_1)\,, \quad (ABl_1l_2\,|\,l_3l_4)\,, \quad (Bl_1l_2l_3\,|\,l_4A)\,,\\
&(ABl_1\,|\,l_2l_3l_4)\,, \quad (Bl_1l_2\,|\,l_3l_4A)\,, \quad (l_1l_2l_3\,|\,l_4AB)\,.
\end{aligned} \tag{5.28}$$

For the subamplitude J, replace A, B by D, C to find $(J_L|J_R)$. (This gives reverse cyclic order on J.) Recall that we are assuming $l_4 \notin I_L,\, J_L$.

In some cases, the spin sum factor for product of diagrams $(I_L|I_R) \times (J_L|J_R)$ can directly be seen to vanish. For instance, if no loop momenta are contained in I_L or J_L, then a row in the matrix (5.26) vanishes, and hence $(\text{s.s.f.})_{I_L,J_L} = 0$. It follows that the diagrams $(AB\,|\,l_1l_2l_3l_4)$ and $(DC\,|\,l_1l_2l_3l_4)$ do not contribute to the spin sum. Another non-contributing case is when I_L and J_L each contain only one loop momentum l_i which is common to both. Then a 2×2 submatrix of (5.26) vanishes, and hence $(\text{s.s.f.})_{I_L,J_L} = 0$. This means that products such as $(Bl_1\,|\,l_2l_3l_4A) \times (DCl_1\,|\,l_2l_3l_4)$ vanish. Finally, it may be noted that if $l_1, l_2, l_3 \in I_L \cap J_L$, then the determinant (5.26) vanishes thanks to the Schouten identity. These observations are general and apply for any number of external legs to reduce the number of terms contributing in the sum over all products of MHV vertex diagrams. For the case of 4 external momenta, the number of contributing diagrams are thus reduced from $9^2 = 81$ to $8^2 - 4 - 4 = 56$.

We have verified numerically for the 4-point amplitude that the sum of all diagrams is independent of both reference spinors.

6 Anti-MHV and anti-NMHV generating functions and spin sums

In the previous section we have evaluated spin sums for unitarity cuts which involved MHV and NMHV subamplitudes.[7] However, the table in figure 3 shows that this is not enough. The unitarity cut at loop order $L = 2, 3$ includes the product of MHV and N^2MHV amplitudes, and N^3MHV is needed at 4-loop order. Our method would then require the generating functions for N^2MHV and N^3MHV amplitudes (see [12] for their construction). However, the situation is also workable if these amplitudes have a small number of external lines. For example, if we are interested in the 4-line cut of a 3-loop 4-point function, then the N^2MHV amplitudes we need are 6-point functions, and these are the complex conjugates of MHV amplitudes, usually called anti-MHV amplitudes. To evaluate their contribution to the intermediate state sum we need an anti-MHV generating function expressed in terms of the original η_{ia} variables, so we can apply our integration techniques. Thus we first describe a general method to construct anti-N^kMHV generating functions from N^kMHV

[7] By conjugation, these results apply quite directly to cuts only involving anti-MHV and anti-NMHV subamplitudes.

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generating functions and use it to find explicit expressions for the anti-MHV and anti-NMHV cases (section 6.1). Then we apply these generating functions to evaluate several examples of unitarity sums in which (N)MHV amplitudes occur on one side of the cut and anti-(N)MHV amplitudes on the other (section 6.2). The most sophisticated example is the intermediate state sum for a 5-line cut of a 4-loop 4-point function.

6.1 Anti-generating functions

An N^{k}MHV amplitude has external states whose η-counts r_i add up to a total of $4(k+2)$. The total η-count is matched in the generating function, which must be a sum of monomials of degree $4(k+2)$ in the variables η_{ia}. The states of the conjugate anti-N^{k}MHV amplitude have η-counts $4-r_i$, so the total η-count for n-point amplitudes is $4(n-(k+2))$. Thus anti-N^{k}MHV generating functions must contain monomials of degree $4(n-(k+2))$. For example, the 3-point anti-MHV generating function has degree 4, and the 6-point anti-MHV and 7-point anti-NMHV cases both have degree 16.

The η-count requirement is nicely realized if we define the anti-N^{k}MHV generating function as [17] the Grassmann Fourier transform of the conjugate of the corresponding N^{k}MHV generating function. Given a set of N Grassmann variables θ_I and their formal adjoints $\bar\theta^I$, the Fourier transform of any function $f(\bar\theta^I)$ is defined as

$$\hat f(\theta_I) \equiv \int d^N\bar\theta \exp(\theta_I\bar\theta^I)\, f(\bar\theta^I)\,. \tag{6.1}$$

Any $f(\bar\theta^I)$ is a sum of monomials of degree $M \le N$, e.g. $\bar\theta^J\cdots\bar\theta^K$, which can be "pulled out" of the integral and expressed as derivatives, viz.

$$\begin{aligned}\int d^N\bar\theta\, \exp(\theta_I\bar\theta^I)\,\bar\theta^J\cdots\bar\theta^K &= (-)^N \frac{\partial}{\partial\theta_J}\cdots\frac{\partial}{\partial\theta_K}\int d^N\bar\theta\, \exp(\theta_I\bar\theta^I)\\ &= \frac{\partial}{\partial\theta_J}\cdots\frac{\partial}{\partial\theta_K}\prod_{I=1}^N\theta_I\\ &= \frac{1}{(N-M)!}\epsilon^{J\ldots KI_{M+1}\ldots I_N}\theta_{I_{M+1}}\ldots\theta_{I_N}\,.\end{aligned} \tag{6.2}$$

The procedure to convert an N^{k}MHV n-point generating function into an anti-N^{k}MHV generating function uses conjugation followed by the Grassmann Fourier transform. The conjugate of any function[8] $f(\langle ij\rangle,[kl],\eta_{ia})$ is defined as $f([ji],\langle lk\rangle,\bar\eta_i^a)$, including reverse order of Grassmann monomials. Evaluation of the Fourier transform

$$\hat f \equiv \int\prod_{i,a} d\bar\eta_i^a\, \exp\left(\sum_{b,j}\eta_{jb}\bar\eta_j^b\right) f([ji],\langle lk\rangle,\bar\eta_i^a)\,, \tag{6.3}$$

is then equivalent to the following general prescription:

1. Interchange all angle and square brackets: $\langle ij\rangle \leftrightarrow [ji]$.

[8] For simplicity it is assumed that the only complex numbers contained in f are the spinor components of $|i\rangle$ and $|i]$.

2. Replace $\eta_{ia} \to \partial_i^a = \frac{\partial}{\partial \eta_{ia}}$.

3. Multiply the resulting expression by $\prod_{a=1}^4 \prod_{i=1}^n \eta_{ia}$ from the right.

We first apply this to find an anti-MHV generating function. We will confirm that the result is correct by showing that it solves the SUSY Ward identities. We will then apply the prescription to find an anti-NMHV generating function.

6.1.1 Anti-MHV generating function

Applied to the conjugate of the MHV generating function (4.1) (with (4.2)), the prescription gives the anti-MHV generating function[9]

$$\bar{F}_n = \frac{1}{\prod_{i=1}^n [i,i+1]} \frac{1}{2^4} \prod_{a=1}^4 \sum_{i,j=1}^n [ij]\, \partial_i^a \partial_j^a\, \eta_{1a} \cdots \eta_{na}\,. \tag{6.4}$$

Evaluating the derivatives as in (6.2) we can write this as

$$\bar{F}_n = \frac{1}{\prod_{i=1}^n [i,i+1]} \frac{1}{(2\,(n-2)!)^4} \prod_{a=1}^4 \sum_{k_1,\ldots,k_n} \epsilon^{k_1 k_2 \cdots k_n}\, [k_1 k_2]\, \eta_{k_3 a} \cdots \eta_{k_n a} \tag{6.5}$$

The sum is over all external momenta $k_i \in \{1,2,\ldots,n\}$.

To confirm that (6.5) is correct we show that $\bar{F}_n$ obeys the SUSY Ward identities and produces the correct all-gluon anti-MHV amplitude $A_n(1^+2^+3^- \ldots n^-)$. This is sufficient because the SUSY Ward identities have a unique solution [11] in the MHV or anti-MHV sectors. Any (anti-)MHV amplitudes can be uniquely written as a spin factor times an n-gluon amplitude. The desired n-gluon amplitude is obtained by applying the product $D_3 \ldots D_n$ of 4th order operators of (2.8) for the $(n-2)$ negative gluons to the generating function (6.5). It is easy to obtain the expected result

$$\begin{aligned} D_3 \cdots D_n \bar{F}_n &= \frac{1}{\prod_{i=1}^n [i,i+1]} \frac{1}{(2\,(n-2)!)^4} \prod_{a=1}^4 \sum_{k_1,k_2} (n-2)!\, \epsilon^{k_1 k_2 34 \ldots n}\, [k_1 k_2] \\ &= \frac{[12]^4}{\prod_{i=1}^n [i,i+1]}\,. \end{aligned} \tag{6.6}$$

The supercharges which act on generating functions are [11]

$$Q^a = \sum_{i=1}^n [i|\, \partial_i^a\,, \qquad \tilde{Q}_a = \sum_{i=1}^n |i\rangle\, \eta_{ia}\,. \tag{6.7}$$

Ward identities are satisfied if Q^a and $\tilde{Q}_a$ annihilate $\bar{F}_n$. Formally this requirement is satisfied by the Grassmann Fourier transform, but we find the following direct proof instructive. We compute

$$Q^a \bar{F}_n \propto \sum_{k_1,k_2,i,k_4,\ldots,k_n} [k_1 k_2][i|\, \epsilon^{k_1 k_2 i k_4 \cdots k_n}\, \eta_{k_4 a} \cdots \eta_{k_n a} \;=\; 0 \tag{6.8}$$

[9] We omit an overall factor $(-1)^n$ in $\bar{F}_n$. This has no consequence for our applications in spin sums.

by the Schouten identity. The argument for $\tilde{Q}_a$ is slightly more involved. First write

$$\tilde{Q}_a \bar{F}_n \propto \sum_{i,k_1,\ldots,k_n} |i\rangle[k_1 k_2]\epsilon^{k_1 k_2 k_3 k_4 \cdots k_n}\, \eta_{ia}\, \eta_{k_3 a} \cdots \eta_{k_n a}\,. \tag{6.9}$$

Note that the product of η's is nonvanishing only when $i \notin \{k_3, \ldots, k_n\}$, i.e. when i is k_1 or k_2. Thus

$$\begin{aligned}\tilde{Q}_a \bar{F}_n &\propto -2 \sum_{k_1,\ldots,k_n} |k_2\rangle[k_2 k_1]\epsilon^{k_1 k_2 k_3 k_4 \cdots k_n}\, \eta_{k_2 a}\, \eta_{k_3 a} \cdots \eta_{k_n a} \\ &= -2(n-2)! \sum_{k_1=1}^{n} \left(\prod_{i \neq k_1} \eta_{ia} \right) \left(\sum_{k_2=1}^{n} |k_2\rangle[k_2 k_1] \right) \;=\; 0\,, \end{aligned} \tag{6.10}$$

due to momentum conservation. For given k_1 the product of $(n-1)$ factors of $\eta_{l_i a}$'s with $l_i \neq k_1$ is the same for all choices of k_2 and it was therefore taken out of the sum over k_2. This completes the proof that (6.5) produces all n-point anti-MHV amplitudes correctly.

For $n = 3$, the generating function (6.5) reduces to the anti-MHV 3-point amplitudes

$$\bar{F}_3 = \frac{1}{[12][23][31]} \prod_{a=1}^{4} \Big([12]\eta_{3a} + [31]\eta_{2a} + [23]\eta_{1a} \Big)\,, \tag{6.11}$$

recently presented in [18].

An alternative form of the anti-MHV generating function can be given for $n \geq 4$. It is more convenient for calculations because it contains the usual $\delta^{(8)}(\sum_{i=1}^{n} |i\rangle\eta_{ia})$ as a factor. The second factor requires the selection of two special lines, here chosen to be 1 and 2. The alternate form reads

$$\bar{F}_n = \frac{1}{\langle 12\rangle^4 \prod_{i=1}^{n}[i, i+1]} \frac{1}{(2(n-4)!)^4} \delta^{(8)}\left(\sum_{i=1}^{n} |i\rangle\eta_{ia} \right) \prod_{a=1}^{4} \sum_{k_3,\ldots,k_n} \epsilon^{12 k_3 \cdots k_n}[k_3 k_4]\eta_{k_5 a} \cdots \eta_{k_n a}\,. \tag{6.12}$$

Arguments very similar to the ones above show that (6.12) satisfies the Ward identities and produces the correct gluon amplitude $A_n(1^- 2^- 3^+ 4^+ 5^- \ldots n^-)$. Since these requirements have a unique realization, the two forms (6.5) and (6.12) must coincide.

For $n = 4, 5$ the anti-MHV generating function (6.12) reduces to the "superamplitudes" recently presented in [17]. It is worth noting that for $n = 4$, any MHV amplitude is also anti-MHV; using momentum conservation it can explicitly be seen that the anti-MHV generating function (6.5), or in the form (6.12), is equal to the MHV generating function for $n = 4$.

6.1.2 Anti-NMHV generating function

Any anti-NMHV n-point amplitude I of the $\mathcal{N} = 4$ theory has an anti-MHV vertex expansion, which is justified by the validity of the MHV vertex expansion of the conjugate NMHV amplitude. For each diagram of the expansion we use the conjugate of the CSW prescription, namely

$$|P_{I_L}] = P_{I_L}|X\rangle\,. \tag{6.13}$$

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This involves a reference spinor $|X\rangle$. The sum of all diagrams is independent of $|X\rangle$.

We will obtain the anti-NMHV generating function by applying the prescription above to the conjugate of the NMHV generator (4.14). This prescription directly gives

$$\bar{\mathcal{F}}_{n,I_L} = \frac{1}{\prod_{i=1}^n [i,i+1]} \overline{W}_{I_L} \frac{1}{2^4} \prod_{a=1}^4 \sum_{i,j\in I} \sum_{k\in I_L} [ij][P_{I_L}\, k]\, \partial_i^a \partial_j^a \partial_k^a\, \eta_{1a} \cdots \eta_{na}\,, \tag{6.14}$$

where $\overline{W}_{I_L}$ is obtained from W_{I_L} in (4.9) by exchanging angle and square brackets.

Carrying out the differentiations and relabeling summation indices gives the desired result:

$$\bar{\mathcal{F}}_{n,I_L} = \frac{1}{\prod_{i=1}^n [i,i+1]} \overline{W}_{I_L} \frac{1}{(2(n-3)!)^4} \prod_{a=1}^4 \sum_{k_1\in I_L} \sum_{k_2,\ldots,k_n\in I} [P_{I_L} k_1][k_2 k_3] \epsilon^{k_1 k_2 \ldots k_n} \eta_{k_4 a} \cdots \eta_{k_n a}\,. \tag{6.15}$$

The factor $1/((n-3)!)^4$ compensates for the overcounting produced by the contraction of the Levi-Civita symbol with the products of η's. The expression (6.15) contains a hidden factor of $\delta^{(8)}(\text{ext})$ and can be written

$$\bar{\mathcal{F}}_{n,I_L} = \frac{\overline{W}_{I_L} \delta^{(8)}\left(\sum_{i=1}^n |i\rangle \eta_{ia}\right)}{(2(n-5)!)^4 \langle 12\rangle^4 \prod_{i=1}^n [i,i+1]} \prod_{a=1}^4 \sum_{k_3\in I_L} \sum_{k_4,\ldots,k_n\in I} \epsilon^{12 k_3 k_4 \ldots k_n} [k_3 P_{I_L}][k_4 k_5] \eta_{k_6 a} \cdots \eta_{k_n a}\,. \tag{6.16}$$

It is not trivial to show that (6.16) follows from (6.15). We present the proof in [12].

In analogy with section 4.3 the universal anti-NMHV generating function is the sum

$$\bar{\mathcal{F}}_n = \sum_{I_L \subset I} \bar{\mathcal{F}}_{n,I_L} \tag{6.17}$$

over all possible diagrams.

One check that this result is correct is to show that the SUSY charges of (6.7) annihilate $\bar{\mathcal{F}}_n$. This check can be carried out, but it is not a complete test that (6.17) is correct because the SUSY Ward identities do not have a unique solution in the NMHV or anti-NMHV sectors. See [19] or [11]. For this reason we show in appendix B that (6.14) is obtained for any diagram starting from the product of anti-MHV generating functions for the left and right subamplitudes. Essentially we obtain (6.14) by the complex conjugate of the process which led from (4.13) to (4.14). It then follows that the application of external line derivatives (of total order $4(n-3)$) to (6.17) produces the correct "anti-MHV vertex expansion" of the corresponding anti-NMHV amplitude.

6.2 Anti-generating functions in intermediate spin sums

With the anti-MHV and anti-NMHV generating functions we can complete the unitarity sums for 3- and 4-loop 4-point amplitudes.

6.2.1 L-loop anti-MHV $\times$ MHV spin sum

Consider the $(L+1)$-line unitarity cut of an L-loop MHV amplitude, as in figure 3. The intermediate spin sum will include a sector where one subamplitude is N^{L-1}MHV and the

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other is MHV. We assume that the full amplitude has a total of 4 external legs, with 2 on each side of the cut, as in figure 5, so the tree subamplitudes have $L+3$ legs. Then[10] the N^{L-1}MHV subamplitude is anti-MHV and we can apply our anti-MHV generating function to obtain the spin sum.

The spin sum is

$$\begin{aligned}\mathcal{F}^{L\text{-loop MHV 4-point}}_{\text{anti-MHV}\times\text{MHV}} &= D_1\cdots D_{L+1}\bar{F}_{L+3}(I)F_{L+3}(J)\\ &= D_1\cdots D_{L+1}\frac{1}{\prod_{i\in I}[i,i+1]}\widetilde{\delta^{(8)}(I)}\frac{1}{\prod_{j\in J}\langle j,j+1\rangle}\delta^{(8)}(J)\,,\end{aligned}\tag{6.18}$$

where

$$\widetilde{\delta^{(8)}(I)} \equiv \frac{1}{\langle l_1 l_2\rangle^4}\frac{1}{(2(L-1)!)^4}\delta^{(8)}(I)\left(\prod_{a=1}^{4}\sum_{k_3,\ldots,k_{L+3}}[k_3k_4]\epsilon^{l_1l_2k_3\cdots k_{L+3}}\eta_{k_5a}\cdots\eta_{k_{L+3}a}\right)\tag{6.19}$$

is obtained from the Fourier transform; we use the form (6.12) for the anti-MHV generating function, selecting the loop momenta l_1 and l_2 as the two special lines. Then, focusing on the spin sum factor only, we have

$$\text{s.s.f.} = D_1\cdots D_{L+1}\,\widetilde{\delta^{(8)}(I)}\,\delta^{(8)}(J)\,.\tag{6.20}$$

Converting the η_1 differentiation to integration we find

$$\begin{aligned}\text{s.s.f.} &= \delta^{(8)}(I+J)D_2\cdots D_{L+1}\left\{\frac{1}{\langle l_1l_2\rangle^4}\frac{1}{(2(L-1)!)^4}\prod_{a=1}^{4}\left(\cdots+\langle l_1l_2\rangle\eta_{2a}+\ldots\right)\right.\\ &\qquad\left.\times\left(\sum_{k_3,\ldots,k_{L+3}}[k_3k_4]\;\epsilon^{l_1l_2k_3\cdots k_{L+3}}\eta_{k_5a}\cdots\eta_{k_{L+3}a}\right)\right\}\\ &= \delta^{(8)}(\text{ext})D_3\cdots D_{L+1}\left\{\frac{1}{(2(L-1)!)^4}\prod_{a=1}^{4}\left(\sum_{k_3,\ldots,k_{L+3}}[k_3k_4]\epsilon^{l_1l_2k_3\cdots k_{L+3}}\eta_{k_5a}\cdots\eta_{k_{L+3}a}\right)\right\}\\ &= [AB]^4\delta^{(8)}(\text{ext})\,.\end{aligned}\tag{6.21}$$

A and B are the external legs on the subamplitude I, c.f. figure 5.

As a simple check that this result is correct, let the legs A and B be positive helicity gluons and take the two other external legs C and D to be negative helicity gluons. Then there is only one term in the spin sum, namely

$$A_{L+3}(A^+,B^+,l_1^-,\ldots,l_{L+1}^-)\;A_{L+3}(C^-,D^-,-l_{L+1}^+,\ldots,-l_1^+)\,,\tag{6.22}$$

whose "spin sum factor" is simply $\langle CD\rangle^4[AB]^4$. This is exactly what our result (6.21) produces when the two 4th order derivative operators D_C and D_D of the external negative helicity gluons are applied.

[10] An $(L+3)$-point anti-MHV amplitude has η-count $4(n-(k+2)) = 4(L+1)$, so it is N^{L-1}MHV.

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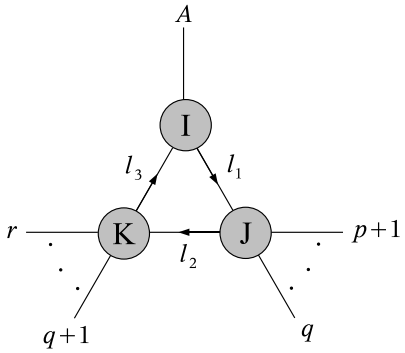

Figure 6. Triple cut of MHV 1-loop amplitude with anti-MHV subamplitude I and MHV subamplitudes J and K.

Rewriting the prefactors to separate the dependence on the loop momenta, the full result for the L-loop $(L+1)$-line cut MHV generating function is then simply

$$\mathcal{F}^{L\text{-loop MHV 4-point}}_{\text{anti-MHV}\times\text{MHV}} = [AB]^4\left([AB][B|l_1|C\rangle\langle CD\rangle\langle D|l_{L+1}|A]\prod_{i=1}^{L}P^2_{i,i+1}\right)^{-1}\delta^{(8)}(\text{ext})\,. \quad (6.23)$$

The result (6.23) of an L-loop calculation is strikingly simple, yet it counts the contributions of states of total η-count $0 \le r \le 8$ distributed arbitrarily on the $L+1$ internal lines in figure 5.

6.2.2 1-loop triple cut spin sum with anti-MHV × MHV × MHV

Consider the triple cut of a 1-loop amplitude. In section 5.1.2 we evaluated a triple cut spin sum assuming that the amplitude was overall NMHV, such that the three subamplitudes were MHV. We now consider the case where the amplitude is overall MHV. The η-count then tells us that $r_I + r_J + r_K = 8 + 4\times 3 = 20$. At least one of the subamplitudes has to be anti-MHV with η-count 4. Thus let us assume I to be anti-MHV and J and K MHV. The result is non-vanishing only if I is a 3-point amplitude, i.e. it has only one external leg, which we will label A. This is illustrated in figure 6.

We evaluate the spin sum using the anti-MHV and MHV generating functions. The expression for the spin sum factor requires some manipulation using momentum conservation, but the final result is simple:

$$\mathcal{F}^{\text{1-loop MHV triple cut}}_{\text{anti-MHV}\times\text{MHV}\times\text{MHV}} = \frac{\langle rA\rangle\langle A,p+1\rangle\langle q,q+1\rangle[l_1A]^4\langle l_1l_2\rangle^4}{[Al_1][l_1l_3][l_3A]\langle l_1,p+1\rangle\langle ql_2\rangle\langle l_2l_1\rangle\langle rl_3\rangle\langle l_3l_2\rangle\langle l_2,q+1\rangle}\mathcal{F}^{\text{tree}}_{\text{MHV}}(\text{ext})\,. \quad (6.24)$$

One simple check of this result is to assign all external states to be gluons, with A and B (on, say, subamplitude J) having negative helicity and the rest positive. Then the spin sum only contains one term, which gives a spin factor $[l_1l_3]^4\langle l_1B\rangle^4\langle l_2l_3\rangle^4$. This must be compared with the result of (6.24) with D_AD_B applied, giving a spin factor $[l_1A]^4\langle l_1l_2\rangle^4\langle AB\rangle^4$.

Momentum conservation on the 3-point subamplitude I gives

$$[l_1 A]^4 \langle l_1 l_2\rangle^4 \langle AB\rangle^4 = [l_3 A]^4 \langle l_3 l_2\rangle^4 \langle AB\rangle^4 = [l_3 l_1]^4 \langle l_3 l_2\rangle^4 \langle l_1 B\rangle^4 \,, \tag{6.25}$$

so the results agree.

6.2.3 4-loop anti-NMHV × NMHV spin sum

The 5-line cut of the 4-point 4-loop amplitude includes an N^{2}MHV × NMHV sector in its unitarity sum. We use notation as in figure 5. The tree subamplitudes are in this case 7-point functions and N^{2}MHV is therefore the same as anti-NMHV. We evaluate the spin sum using the NMHV and anti-NMHV generating functions.

Consider the anti-NMHV$_7(I)$ × NMHV$_7(J)$ sector of the 5-line cut of the 4-loop 4-point amplitude. The intermediate state sum is straightforward to evaluate using the anti-NMHV generating function in the form (6.16), choosing the lines l_1 and l_2 as the special lines 1 and 2. The result of the intermediate spin sum is then

$$\mathcal{F}^{\text{4-loop MHV 5-line cut}}_{\text{anti-NMHV}_7\times\ \text{NMHV}_7} = \delta^{(8)}(\text{ext}) \frac{\sum_{I_L,J_L} \overline{W}_{I_L} W_{J_L} (\text{s.s.f})_{I_L,J_L}}{\langle l_1 l_2\rangle^4 \big(\prod_{i\in I}[i,i+1]\big)\big(\prod_{j\in J}\langle j,j+1\rangle\big)} \tag{6.26}$$

where

$$\begin{aligned}(\text{s.s.f})_{I_L,J_L} = \Bigg\{ \sum_{j=3}^{5} \Big[\delta_{l_j\in J_L} \langle l_j\, P_{J_L}\rangle \langle l_1 l_2\rangle + \delta_{l_1\in J_L} \langle l_1\, P_{J_L}\rangle \langle l_2 l_j\rangle + \delta_{l_2\in J_L} \langle l_2\, P_{J_L}\rangle \langle l_j l_1\rangle \Big] \\ \times \Big[\delta_{l_j\in I_L} [l_j\, P_{I_L}][A\, B] + \delta_{A\in I_L} [A\, P_{I_L}][B\, l_j] + \delta_{B\in I_L} [B\, P_{I_L}][l_j\, A] \Big] \Bigg\}^4 \end{aligned} \tag{6.27}$$

The sum $\sum_{I_L,J_L}$ is over all 13 anti-MHV and MHV vertex diagrams in the expansions of the subamplitudes I and J. We have checked numerically that the cut amplitude generating function is independent of the two reference spinors $|X_I\rangle$ and $|X_J]$ from the CSW prescription of $|P_{I_L}]$ and $|P_{J_L}\rangle$.

The complete spin sum for this cut of the 4-loop 4-point amplitude contains the four contributions listed in the table in figure 3. The anti-MHV × MHV contribution is obtained as the $L = 4$ case of (6.23) and we have here presented the result for the anti-NMHV × NMHV spin sum. The MHV × anti-MHV and NMHV × anti-NMHV contributions are obtained directly from these results.

6.2.4 Other cuts of the 4-loop 4-point amplitude

The full 4-loop calculation requires the study of unitarity cuts in which a 6-point subamplitude appears with all 6 lines internal and cut. The simplest case is that of a 4-point function, hence overall MHV, which can be expressed as the product $(2 \to 3)(3 \to 3)(3 \to 2)$ of 3 sub-amplitudes. See figure 7. The spin sum requires integration over the $6 \times 4 = 24$ η_{ia} variables of the internal lines, and each term contains 8 of the 16 Grassmann variables $\eta_{Aa}.\eta_{Ba}, \eta_{Ca}, \eta_{Da}$ associated with the external states. Thus, before any integrations, we are

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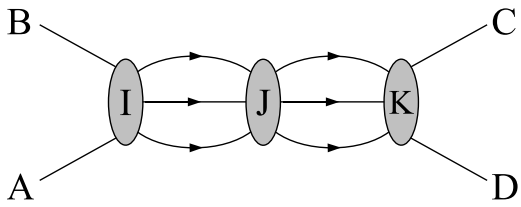

Figure 7. A unitarity cut of diagrams that contribute to the 4-point MHV amplitude at 4 loops. Note that subamplitude J only connects to internal lines.

dealing with a product of generating functions containing monomials of degree $8+24=32$. The full unitarity sum contains several sectors in which the 32 η's are split as

$$\begin{array}{rcl} 8+16+8 & \leftrightarrow & I\times J\times K \;=\; \mathrm{MHV}_5\times\;\overline{\mathrm{MHV}}_6\;\times\mathrm{MHV}_5 \\ 12+8+12 & & \qquad\quad\overline{\mathrm{MHV}}_5\times\;\mathrm{MHV}_6\;\times\overline{\mathrm{MHV}}_5 \\ 8+12+12 & & \qquad\quad\mathrm{MHV}_5\times\mathrm{NMHV}_6\times\overline{\mathrm{MHV}}_5 \\ 12+12+8 & & \qquad\quad\overline{\mathrm{MHV}}_5\times\mathrm{NMHV}_6\times\mathrm{MHV}_5. \end{array} \tag{6.28}$$

We have carried out each of these spin sums explicitly. The first two cases are related to each other by conjugation (including conjugation of the external states). The last two are related by interchanging I and K and relabeling the internal momenta accordingly. The 6-point NMHV amplitude can also be regarded as anti-NMHV. We have calculated the spin sums in both ways, using the NMHV and anti-NMHV generating function for J. Different diagrams contribute in these calculations, but numerically the results agree (and they are independent of the reference spinors).

7 Valid 2-line shifts for any $\mathcal{N}=4$ SYM amplitude

In this section we turn our attention to 2-line shifts which give recursion relations of the BCFW type. We examine the behavior of a general $\mathcal{N}=4$ SYM tree level amplitude

$$A_n(1\ldots i\ldots j\ldots n) \tag{7.1}$$

under a 2-line shift of type $[i,j\rangle$, i.e.

$$|\tilde{i}] = |i] + z|j]\,, \qquad |\tilde{i}\rangle = |i\rangle\,, \qquad |\tilde{j}] = |j]\,, \qquad |\tilde{j}\rangle = |j\rangle - z|i\rangle\,, \tag{7.2}$$

with $i\neq j$. We will show that for *any* amplitude A_n with $n>4$, we can find a valid shift $[i,j\rangle$ such that the amplitude vanishes at least as fast as $1/z$ for large z. This implies that there is a valid BCFW recursion relation for any tree amplitude in $\mathcal{N}=4$ SYM.

The strategy of our proof is the following. In [7] it was shown that amplitudes A_n vanish at large z under a shift $[i^-,j\rangle$ if line i is a negative helicity gluon. We extend this result using supersymmetric Ward identities and show that amplitudes vanish at large z under any shift $[i,j\rangle$ in which the SU(4) indices carried by the particle on line j are a

subset of the SU(4) indices of particle i. We then show that such a choice of lines i and j exists for all non-vanishing amplitudes A_n with $n \geq 4$, except for some pure scalar 4- and 6-point amplitudes. The 4-point amplitude is MHV hence determined by the SUSY Ward identities. We then analyze the scalar 6-point amplitudes explicitly and find that there exist valid shifts $[i, j\rangle$ under which they vanish at large z.

7.1 Large z behavior from Ward identities

For an $\mathcal{N} = 4$ n-point tree level amplitude A_n of the form (7.1) it was shown in [7] that

$$\begin{gathered} A_n(1 \dots i^- \dots j \dots n) \sim O(z^{-1}) \ \text{ under a } [i^-, j\rangle \text{ shift} \\ \text{if } i \text{ is a negative helicity gluon, line } j \text{ arbitrary}. \end{gathered} \tag{7.3}$$

Now consider any amplitude A_n which has two lines i and j such that the SU(4) indices of line j are a subset of the SU(4) indices of line i. We will prove that the amplitude vanishes at large z under the BCFW $[i, j\rangle$-shift given in (7.2). Specifically we will show that

$$\begin{gathered} A_n(1 \dots i \dots j \dots n) \sim O(z^{-1}) \ , \ \text{ or better, under the shift } [i, j\rangle \\ \text{if all SU(4) indices of } j \text{ are also carried by } i\,. \end{gathered} \tag{7.4}$$

Let r_i be the number of SU(4) indices carried by line i. We will show (7.4) by (finite, downward) induction on r_i. For $r_i = 4$ particle i is a negative helicity gluon and the statement (7.4) reduces to (7.3) which was proven in [7]. Assume now that (7.4) is true for all amplitudes with $r_i = \bar{r}$ for some $1 \leq \bar{r} \leq 4$. Consider any amplitude A_n which has $r_i = \bar{r} - 1 < 4$ and in which the SU(4) indices of particle j are a subset of the indices of particle i. We write this amplitude as a correlation function

$$A_n(1 \dots i \dots j \dots n) = \langle \mathcal{O}(1) \dots \mathcal{O}(i) \dots \mathcal{O}(j) \dots \mathcal{O}(n) \rangle\,. \tag{7.5}$$

Pick an SU(4) index a which is *not* carried by line i. Such an index exists because $r_i < 4$. There exists an operator $\mathcal{O}^a(i)$ such that

$$[\tilde{Q}_a, \mathcal{O}^a(i)] = \langle \epsilon i \rangle\, \mathcal{O}(i)\,. \qquad \text{(no sum)} \tag{7.6}$$

By assumption, the SU(4) index a is also not carried by line j, so $[\tilde{Q}_a, \mathcal{O}(j)] = 0$. We can now write a Ward identity based on the index a as follows:

$$\begin{aligned} 0 &= \big\langle \big[\tilde{Q}_a\,, \mathcal{O}(1) \dots \mathcal{O}^a(i) \dots \mathcal{O}(j) \dots \mathcal{O}(n)\big] \big\rangle \\ &= \langle \epsilon i \rangle \big\langle \mathcal{O}(1) \dots \mathcal{O}(i) \dots \mathcal{O}(j) \dots \mathcal{O}(n) \big\rangle + \big\langle \big[\tilde{Q}_a\,, \mathcal{O}(1) \dots \big]\, \mathcal{O}^a(i) \dots \mathcal{O}(j) \dots \mathcal{O}(n) \big\rangle \\ &\quad + \big\langle \mathcal{O}(1) \dots \mathcal{O}^a(i)\, \big[\tilde{Q}_a\,, \dots\big]\, \mathcal{O}(j) \dots \mathcal{O}(n) \big\rangle + \big\langle \mathcal{O}(1) \dots \mathcal{O}^a(i) \dots \mathcal{O}(j)\, \big[\tilde{Q}_a\,, \dots \mathcal{O}(n)\big] \big\rangle\,. \end{aligned} \tag{7.7}$$

Let us choose $|\epsilon\rangle \sim |\ell\rangle \neq |i\rangle, |j\rangle$. Then the first term on the right hand side is the original amplitude (7.5), multiplied by a non-vanishing factor $\langle \ell i \rangle$, which does not shift under the $[i, j\rangle$-shift. The remaining three terms on the right hand side of (7.7) all involve the operators $\mathcal{O}^a(i)$ and $\mathcal{O}(j)$. The number of SU(4) indices carried by $\mathcal{O}^a(i)$ is $r_i + 1 = \bar{r}$, and therefore, by the inductive assumption, each of the remaining amplitudes fall off at least

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as fast as $1/z$ under the $[i,j\rangle$-shift. They are multiplied by angle brackets of the form $\langle \ell k\rangle$ with $k \neq i,j$. These angle brackets do not shift. Thus the last three terms on the right side of (7.7) go as $1/z$ or better for large z. We conclude that the amplitude $A_n(1\dots i\dots j\dots n)$ also vanishes at least as $1/z$ for large z under the $[i,j\rangle$-shift. This completes the inductive step and proves (7.4).

Our result implies, in particular, that any shift $[i,j^+\rangle$ gives a $1/z$ falloff for any state i. This is because a positive helicity gluon j^+ carries no SU(4) indices, and the empty set is a subset of any set.

7.2 Existence of a valid 2-line shift for any amplitude

We have proven the existence of a valid recursion relation for any amplitude which admits a shift of the type (7.4). Let us examine for which amplitudes such a shift is possible. In other words, we study which amplitudes contain two lines i and j such that the SU(4) indices carried by line j are a subset of the indices carried by line i. For n-point functions with $n \geq 4$ we find:

- *Any amplitude which contains one or more gluons admits a valid shift.* If the amplitude contains a negative helicity gluon we pick this particle as line i. On the other hand, if the amplitude contains a positive helicity gluon we pick the positive helicity gluon as line j. Independent of the choice of particle for the other shifted line, (7.4) guarantees that the amplitude vanishes for large z under the shift $[i,j\rangle$.

- *Any amplitude with one or more positive helicity gluinos admits a valid shift.* We pick the positive helicity gluino as line j. Denote the SU(4) index carried by this gluino by a. If no other line carries this index a, the amplitude vanishes. Thus in a non-vanishing amplitude there must be at least one other line $i \neq j$ which carries the index a. As line j does not carry indices other than a, we can apply (7.4) and conclude that the amplitude falls off at least as $1/z$ under the shift $[i,j\rangle$.

- *Any amplitude with one or more negative helicity gluinos admits a valid shift.* This proof is the SU(4) conjugate version of the proof above. Now we pick the negative helicity gluino as line i and denote the SU(4) index which is *not* carried by this gluino by a. If all other lines carry this index a, the amplitude vanishes. To see this, pick any other line $k \neq i$. The operator $\mathcal{O}^a(k)$ on this line carries the index a, so there exists an operator $\mathcal{O}(k)$ which satisfies

$$[Q^a, \mathcal{O}(k)] = [\epsilon k]\, \mathcal{O}^a(k)\,. \tag{7.8}$$

  Picking $|\epsilon] \sim |i]$ we obtain

$$0 = \big\langle \big[Q^a\,, \mathcal{O}(1)\dots\mathcal{O}(i)\dots\mathcal{O}(k)\dots\mathcal{O}(n)\big]\,\big\rangle \sim [ik]\big\langle \mathcal{O}(1)\dots\mathcal{O}(i)\dots\mathcal{O}^a(k)\dots\mathcal{O}(n)\big\rangle\,. \tag{7.9}$$

  As $i \neq k$, $[ik]$ is non-vanishing, and we conclude that the amplitude must vanish.

  Thus in a non-vanishing amplitude there must be at least one other line $j \neq i$ which does not carry the index a. As line i carries all indices except for a, we can apply (7.4) and again conclude that the amplitude falls off at least as $1/z$ under the $[i,j\rangle$-shift.

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- *Any pure scalar amplitude A_n with $n > 6$ admits a valid shift.* In a pure scalar amplitude each particle carries two SU(4) indices. There are $\binom{4}{2} = 6$ different combinations of indices possible, corresponding to the six distinct scalars of $\mathcal{N} = 4$ SYM. Thus any pure scalar amplitude A_n with $n > 6$ must have at least two lines i and j with the same particle and thus with coinciding SU(4) indices. Using (7.4) we find that the amplitude vanishes for large z under the $[i, j\rangle$-shift.

We are left to analyze pure scalar amplitudes with $n \leq 6$. Amplitudes containing two identical scalars admit a valid shift by (7.4). Thus we need only check amplitudes which involve distinct scalars:

- $n = 4$: there are two types of inequivalent 4-point amplitudes with four distinct scalars. The first type is the constant amplitude $\langle A^{12}A^{23}A^{34}A^{41}\rangle = 1$, and SU(4) equivalent versions thereof. The only contribution to this amplitude is from the 4-scalar interaction in the Lagrangian. Clearly, it does not a have any good 2-line shifts, but since it is MHV it can be determined uniquely from SUSY Ward identities. The second type of amplitudes are SU(4) equivalent versions of

$$\langle A^{12}A^{34}A^{23}A^{41}\rangle = \frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle}\,. \tag{7.10}$$

  For example, this amplitude vanishes at large z under a $[1, 3\rangle$-shift, and thus admits a valid recursion relation.

- $n = 5$: by SU(4) invariance, there are no non-vanishing 5-point functions with 5 distinct external scalars.

- $n = 6$: we perform explicit checks of the pure scalar amplitudes A_6 in which the external particles are precisely the six distinct scalars of the theory, i.e. the amplitudes involving the particles A^{12}, A^{13}, A^{14}, A^{23}, A^{24}, and A^{34}. We find that all possible permutations of the color ordering of the six scalars give amplitudes which fall off as $1/z$ under a shift $[i, i + 3\rangle$ for some choice of line i. This is done by explicitly computing each amplitude using the NMHV generating function, whose validity was proven in section 3, and then numerically testing the $[i, i + 3\rangle$-shifts for different choices of line i.

We conclude that for any $\mathcal{N} = 4$ SYM amplitude with $n > 4$ there exists at least one choice of lines i and j such that under a $[i, j\rangle$-shift

$$A_n(1 \ldots \tilde{i} \ldots \tilde{j} \ldots n) \to 0 \quad \text{as } z \to \infty\,. \tag{7.11}$$

The results also holds for $n = 4$, with the exception of the 4-scalar amplitude mentioned above.

The input needed for our proof of (7.11) was the result [7] that a $[-, j\rangle$-shift gives a $1/z$-falloff (or better) for any state j. In $\mathcal{N} = 4$ SYM, the validity of a $[-, j\rangle$-shift can also be derived from the validity of shifts of type $[-, -\rangle$ using supersymmetric Ward identities. Thus we could have started with less: to derive (7.11) it is sufficient to know that amplitudes vanish at large z under any $[-, -\rangle$-shift.

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8 Summary and Discussion

In this paper we have explored the validity and application of recursion relations for n-point amplitudes with general external states in $\mathcal{N} = 4$ SYM theory. We now summarize our results, discuss some difficulties which limit their extension to $\mathcal{N} = 8$ supergravity, and comment on some recent related papers.

1. We were especially concerned with recursion relations following from 3-line shifts because these give the most convenient representations for NMHV amplitudes, namely the MHV vertex expansion. We were motivated by the fact that these representations are useful in the study of multi-loop amplitudes in $\mathcal{N} = 4$ SYM, and it is important [20] to know that they are valid. The expansion can be derived using analyticity in the complex variable z of the 3-line shift (3.3) if the shifted amplitude vanishes as $z \to \infty$. We proved that this condition holds *if the 3 shifted lines carry at least one common* SU(4) *index*. SU(4) invariance guarantees that at least one such shift such exists for any NMHV amplitude. For shifts with no common index, there are examples of amplitudes which do not vanish at large z and other examples which do. So the common index criterion is sufficient but not always necessary.

2. We reviewed the structure of the MHV vertex expansion in order to emphasize properties which are important for our applications. A valid 3-line shift, which always exists, is needed to derive the expansion but there is no trace of that shift in the final form of the expansion. For most amplitudes there are several valid shifts, and each leads to the same expansion, which is therefore unique. The main reason for this is that the MHV subdiagrams depend only on holomorphic spinors $|i\rangle$ and $|P_I\rangle$ of the external and internal lines of a diagram. These are not shifted, since the shift affects only the anti-holomorphic spinors $|m_i]$ of the 3 shifted external lines, m_1, m_2, m_3. These desireable properties allow the definition of a universal NMHV generating function which describes all possible n-point processes. This generating function is written as a sum of an "over-complete" set of diagrams which can potentially contribute. Particular amplitudes are obtained by applying a 12th order differential operator in the Grassmann variables of the generating function, and each diagram then appears multiplied by its spin factor. The spin factor vanishes for diagrams which do not contribute to the MHV vertex expansion of a given amplitude. What remains are the diagrams, each in correct form, which actually contribute to the expansion.

3. In [11] it was shown how to use the MHV generating function to carry out the intermediate spin sums in the unitarity cuts from which loop amplitudes are constructed from products of trees. In this paper we have used Grassmann integration to simplify and generalize the previously treated MHV level sums, and we have computed several new examples of sums which require the NMHV generating function on one or both sides of the unitarity cut. The external states in the cut amplitudes are arbitrary and we were able to describe this state dependence with new generating functions.

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4. It is well known that the full set of amplitudes in $\mathcal{N} = 4$ SYM theory includes the anti-MHV sector. This contains the n-gluon amplitude in helicity configuration $A_n(++--\cdots--)$ and all others related by SUSY transformations. Each anti-MHV amplitude is the complex conjugate of an MHV amplitude, but this description is not well suited to the evaluation of unitarity sums. Similar remarks apply to anti-NMHV amplitudes which include $A_n(+++--\cdots-)$ and others related by SUSY. For this reason we developed generating functions for anti-MHV and anti-NMHV n-point amplitudes. We used a systematic prescription to convert any generating function to the conjugate generating function by conjugation of brackets and a simple transformation to a new function of the same Grassmann variables η_{ia}. We then performed 3- and 4-loop unitarity sums in which anti-MHV or anti-NMHV amplitudes appear on one side of the cut and MHV or NMHV on the other side.

5. Our study of the large z behavior of NMHV amplitudes required starting with a concrete representation for them on which we could then perform a 3-particle shift. We used the BCFW recursion relation which is based on a 2-line shift. It was very recently shown in [7] that such a recursion relation is valid for any amplitude in $\mathcal{N} = 4$ SYM which contains at least one negative helicity gluon. Using SUSY Ward identities we were able to remove this restriction. The BCFW recursion relation is valid for all amplitudes.[11]

It is natural to ask whether the properties found for recursion relations and generating functions in $\mathcal{N} = 4$ SYM theory are true in $\mathcal{N} = 8$ supergravity. Unfortunately the answer is that not all features carry over at the NMHV level. One complication is that the shifted MHV subamplitudes which appear in the MHV vertex expansion involve the shifted spinors $|m_i]$, so the expansion is no longer shift independent or unique. Valid expansions can be established for many 6-point NMHV amplitudes, but it is known [11] that there are some amplitudes which do not vanish at large z for any 3-line shift. In these cases one must fix the reference spinor $|X]$ such that the $O(1)$ term at $z \to \infty$ vanishes in order to obtain a valid MHV vertex expansion.

Concerning the 2-line shift recursion relations, there are amplitudes in $\mathcal{N} = 8$ SG which do not admit any valid 2-line shifts. One example is the 6-scalar amplitude $\langle \phi^{1234}\phi^{1358}\phi^{1278}\phi^{5678}\phi^{2467}\phi^{3456}\rangle$. No choice of two lines satisfies the index subset criteria needed in section 7 above, and a numerical analysis shows that there no valid 2-line shifts [11], contrary to the analogous $\mathcal{N} = 4$ SYM cases.

We would like to mention several very recent developments which provide, in effect, new versions of generating functions for amplitudes in $\mathcal{N} = 4$ SYM theory.

The paper [21] presents expressions for tree and loop amplitudes based on the dual conformal symmetry [22, 23]. This symmetry can be proven at tree level using an interesting new recursion relation [18] for amplitudes with general external states. The formula for NMHV tree amplitudes in [21] has the feature that it does not contain the arbitrary reference spinor that characterizes the MHV vertex expansions of [1]. Dual conformal sym-

[11] Except for one particular 4-scalar amplitude which is constant and thus inert under all shifts.

metry appears to be a fundamental and important property of on-shell amplitudes, but the presence of a reference spinor may well be an advantage. Indeed, MHV vertex expansions provide expressions for amplitudes that are quite easy to implement in numerical programs, and the test that the full amplitudes are independent of the reference spinor is extremely useful in practical applications.

The paper [17] has several similarities with our work. They use the same SUSY generators devised in [11] and used here, the MHV generating function of [9] is common, and for $n = 3, 4, 5$ the anti-MHV generating functions coincide. For $n \geq 6$ there are apparent differences in the representation of NMHV amplitudes, since the MHV vertex expansion is not directly used in [17, 21] and there is no reference spinor. It could be instructive to explore the relation between these representations. In [17] the application of generating functions to double and triple cuts of 1-loop amplitudes initiated in [11] and studied above are extended to quadruple cuts with interesting results for the box coefficients which occur.

The very new paper [24] uses the fermion coherent state formalism to derive a new type of tree-level recursion relation for the entire set of $\mathcal{N} = 4$ amplitudes. There are many other intriguing ideas to study here.

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A 3-line shifts of NMHV amplitudes with a negative helicity gluon

In section 3 we considered NMHV n-point amplitudes $A_n(1^-, \ldots, m_2, \ldots, m_3, \ldots, n)$, with particle 1 a negative helicity gluon and m_2 and m_3 sharing at least one common SU(4) index. We claimed that one always obtains a valid MHV vertex expansion from the 3-line shift $[1, m_2, m_3|$. In this appendix we provide the detailed proof of this claim. As a starting point we use the result [7] that a $[1^-, \ell\rangle$-shift of any tree amplitude of $\mathcal{N} = 4$ SYM falls off at least as $1/z$ for large z, for any choice of particle $\ell \neq 1$. The $[1^-, \ell\rangle$-shift therefore gives a valid recursion relation without contributions from infinity.

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Our strategy is then as follows. In section A.1, we first express the NMHV amplitude A_n in terms of the recursion relation following a $[1^-, \ell\rangle$ shift and examine the resulting diagrams. In section A.2, we perform a secondary $[1, m_2, m_3|$ shift on the vertex expansion resulting from the $[1^-, \ell\rangle$ shift. We pick particle ℓ for the first shift such that it is non-adjacent to lines m_2 and m_3. This is always possible for $n \geq 7$ (except for one special case at $n = 7$ which we examine separately in section A.4). We show that for large z each diagram in the $[1^-, \ell\rangle$-expansion falls off at least as $1/z$ under the $[1, m_2, m_3|$-shift, *provided* all NMHV amplitudes A_{n-1} fall off as $1/z$ under a 3-line shift of this same type. This allows us to prove the falloff under the shift inductively in section A.3. In section A.5 we explicitly verify the falloff for $n = 6$ which validates the induction and completes the proof.

A.1 Kinematics and diagrams of the $[1^-, \ell\rangle$ shift

The $[1^-, \ell\rangle$-shift is defined as

$$|\tilde{1}] = |1] + z|\ell], \qquad |\tilde{1}\rangle = |1\rangle, \qquad |\tilde{\ell}] = |\ell], \qquad |\tilde{\ell}\rangle = |\ell\rangle - z|1\rangle, \tag{A.1}$$

where particle 1 is a negative helicity gluon, while line ℓ is arbitrary. Consider a diagram of the $[1^-, \ell\rangle$-expansion with internal momentum $\tilde{P}_{1K} = \tilde{1} + K$. The condition that the internal momentum is on-shell fixes the value of z at the pole to be $z_{1K} = \frac{P_{1K}^2}{\langle 1|K|\ell]}$, so that the shifted spinors at the pole are

$$|\tilde{1}] \;=\; |1] + \frac{P_{1K}^2}{\langle 1|K|\ell]}|\ell], \qquad |\tilde{\ell}\rangle \;=\; |\ell\rangle - \frac{P_{1K}^2}{\langle 1|K|\ell]}|1\rangle. \tag{A.2}$$

At the pole, the internal momentum $\tilde{P}_{1K}$ can be written as

$$(\tilde{P}_{1K})^{\dot{\alpha}\beta} = \frac{P_{1K}|\ell]\,\langle 1|P_K}{\langle 1|K|\ell]}. \tag{A.3}$$

This expression factorizes because $\tilde{P}_{1K}$ is null. It is then convenient to define spinors associated with $\tilde{P}_{1K}$ as

$$|\tilde{P}_{1K}\rangle = \frac{P_{1K}|\ell]\,\langle 1\ell\rangle}{\langle 1|K|\ell]}, \qquad [\tilde{P}_{1K}| = \frac{\langle 1|P_K}{\langle 1\ell\rangle}. \tag{A.4}$$

For future reference, we also record a selection of spinor products:

$$\langle\tilde{1}\tilde{\ell}\rangle = \langle 1\ell\rangle, \qquad \langle\tilde{1}\tilde{P}_{1K}\rangle = \langle 1\ell\rangle, \qquad \langle\tilde{\ell}\tilde{P}_{1K}\rangle = -\frac{\langle 1\ell\rangle\, P_{1K\ell}^2}{\langle 1|K|\ell]}, \tag{A.5}$$

$$[\tilde{1}\tilde{\ell}] = [1\ell], \qquad [\tilde{1}\tilde{P}_{1K}] = -\frac{K^2}{\langle 1\ell\rangle}, \qquad [\tilde{\ell}\tilde{P}_{1K}] = -\frac{\langle 1|K|\ell]}{\langle 1\ell\rangle}. \tag{A.6}$$

We write the diagrams resulting from the $[1^-, \ell\rangle$ shift such that line 1 is always on the Left sub-amplitude L and line ℓ on the Right sub-amplitude R. We denote the total number of legs on the L (R) subamplitude by n_L (n_R). Applied to the n-point amplitude A_n, we have $n_L + n_R = n + 2$.

We can use kinematics to rule out the following classes of L $\times$ R diagrams:

- There are *no* MHV $\times$ MHV diagrams with $n_R = 3$.

  Proof: on the R side we would have a 3-vertex with lines ℓ, P_{1K} and one more line $y \in \{\ell - 1, \ell + 1\}$. The R vertex is MHV when $r_\ell + r_y + r_P = 8$, which requires $r_\ell + r_y \geq 4$. The value of the R subamplitude is fixed by "conformal symmetry" (see sec 5 of [11])

$$A_R = \langle y\, \tilde{\ell}\rangle^{r_y + r_\ell - 5} \langle y\, \tilde{P}_{1K}\rangle^{3 - r_\ell} \langle \tilde{\ell}\, \tilde{P}_{1K}\rangle^{3 - r_y} . \tag{A.7}$$

  Upon imposing momentum conservation $P_{1K} = -p_\ell - p_y$, short calculations yield

$$\langle y\, \tilde{\ell}\rangle = \langle y\, \tilde{P}_{1K}\rangle = \langle \tilde{\ell}\, \tilde{P}_{1K}\rangle = 0 . \tag{A.8}$$

  So all three angle brackets entering A_R vanish. Since A_R has one more angle bracket in the numerator than in the denominator, the amplitude vanishes in the limit where we impose momentum conservation.

- There are *no* anti-MHV $\times$ NMHV diagrams.

  Proof: on the L side we would have a 3-vertex with lines 1, $-P_{1K}$ and one more line $x \in \{2, n\}$. For this vertex to be anti-MHV we need $r_1 + r_x + r_P = 4$, and since $r_1 = 4$, this diagram only exists if line x is a positive helicity gluon, i.e. $r_x = 0$. The value of this subamplitude is

$$A_L = \frac{[x\, \tilde{P}_{1x}]^3}{[\tilde{1}x][\tilde{1}\tilde{P}_{1x}]} , \tag{A.9}$$

  but using momentum conservation we find that each square bracket vanishes:

$$[x\, \tilde{P}_{1x}] = [\tilde{1}x] = [\tilde{1}\tilde{P}_{1x}] = 0 . \tag{A.10}$$

  As A_L has more square brackets in the numerator than in the denominator we conclude that the L subamplitude vanishes.

Thus only the following two types of diagrams contribute to the recursion relation:

Type A: MHV $\times$ MHV diagrams with $n_L \geq 3$ and $n_R \geq 4$.

Type B: NMHV $\times$ anti-MHV diagrams with $n_L = n - 1$ and $n_R = 3$.

We have thus obtained a convenient representation of the amplitude $A_n(1^-, \ldots, m_2, \ldots, m_3, \ldots, n)$ using the 2-line shift $[1^-, \ell\rangle$. We will now use this representation of the amplitude to examine its behavior under a 3-line shift.

A.2 The secondary $|1, m_2, m_3]$ shift

We now act with the 3-line shift $|1, m_2, m_3]$ whose validity we want to prove. The shift $[1, m_2, m_3|$ is defined as

$$\begin{aligned} |\hat{1}] &= |1] + z\langle m_2 m_3\rangle |X] , \\ |\hat{m}_2] &= |m_2] + z\langle m_3 1\rangle |X] , \\ |\hat{m}_3] &= |m_3] + z\langle 1 m_2\rangle |X] . \end{aligned} \tag{A.11}$$

By assumption, the lines m_2 and m_3 have at least one SU(4) index in common. Such a choice is possible for any NMHV amplitude. Up to now, we have not constrained our choice of line ℓ for the primary shift. It is now convenient to choose an $\ell \notin \{m_2, m_3\}$ *which is not adjacent* to either m_2 or m_3. This is always possible for $n \geq 7$, except for one special case with $n = 7$ which we examine separately below. We will now show that under the shift (A.11), amplitudes vanish at least as $1/z$ for large z, *provided* this falloff holds for all NMHV amplitudes with $n-1$ external legs under the same type of shift. This will be the inductive step of our proof.

The action of the shift on the recursion diagrams depends on how m_2 and m_3 are distributed between the L and R subamplitudes. We need to consider three cases: $m_2, m_3 \in L$, $m_2, m_3 \in R$ and $m_2 \in R, m_3 \in L$ (or, equivalently, $m_2 \in L, m_3 \in R$).

Case I: $m_2, m_3 \in R$. The legs on the R subamplitude include $\ell, m_2, m_3, \tilde{P}_{1K}$ as well as at least one line separating ℓ from $m_{2,3}$, so $n_R \geq 5$. Hence the diagram must be of type A: MHV $\times$ MHV.

Since $m_2, m_3 \notin K$, the angle-square bracket $\langle 1|K|\ell]$ is unshifted, but

$$\hat{P}_{1K}^2 = P_{1K}^2 - z\langle m_2 m_3\rangle \langle 1|K|X], \tag{A.12}$$

and therefore

$$\begin{aligned} |\hat{\tilde{1}}] &= |\tilde{1}] + z\langle m_2 m_3\rangle \frac{\langle 1|\ell|X]}{\langle 1|K|\ell]}|\tilde{P}_{1K}], \\ |\hat{\tilde{P}}_{1K}\rangle &= |\tilde{P}_{1K}\rangle - z\langle m_2 m_3\rangle \frac{\langle 1|\ell|X]}{\langle 1|K|\ell]}|\tilde{1}\rangle, \\ |\hat{\tilde{\ell}}\rangle &= |\tilde{\ell}\rangle + z\langle m_2 m_3\rangle \frac{\langle 1|K|X]}{\langle 1|K|\ell]}|\tilde{1}\rangle. \end{aligned} \tag{A.13}$$

while $|\hat{\tilde{P}}_{1K}] = |\tilde{P}_{1K}]$. For arbitrary external lines $a \notin \{\hat{\tilde{\ell}}, \tilde{1}\}$ one can check that

$$\langle a\hat{\tilde{P}}_{1K}\rangle \sim O(z), \qquad \langle a\hat{\tilde{\ell}}\rangle \sim O(z). \tag{A.14}$$

The remaining angle brackets shift as follows:

$$\langle \hat{\tilde{\ell}}\hat{\tilde{P}}_{1K}\rangle \sim O(z), \qquad \langle \tilde{1}\hat{\tilde{P}}_{1K}\rangle \sim O(1), \qquad \langle \tilde{1}\hat{\tilde{\ell}}\rangle \sim O(1), \tag{A.15}$$

while all other angle brackets are $O(1)$.

We can now examine the effect of the $|1, m_1, m_2]$ shift on the MHV $\times$ MHV diagram:

- A_L: on the L subamplitude, only $|\hat{\tilde{1}}]$ and $|\hat{\tilde{P}}_{1K}\rangle$ shift. The shift is a (rescaled) $[\tilde{1}^-, \tilde{P}_{1K}\rangle$-shift and thus A_L falls off at least as $1/z$ for large z by the results of [7].
- The propagator gives a factor $1/z$.
- A_R: since line $\tilde{1}$ belongs to the L subamplitude, $\langle \tilde{1}\hat{\tilde{P}}_{1K}\rangle$ and $\langle \tilde{1}\hat{\tilde{\ell}}\rangle$ do not appear in A_R and it thus follows from (A.14) and (A.15) that all angle brackets in A_R which involve $\hat{\tilde{P}}_{1K}$ or $\hat{\tilde{\ell}}$ are $O(z)$ under the shift. The numerator of A_R consists of four

angle brackets and grows at worst as z^4 for large z. If $\tilde{P}_{1K}$ and $\tilde{\ell}$ are consecutive lines in the R subamplitude, then the denominator of A_R contains three shifted angle bracket and therefore goes as z^3. Otherwise, the denominator contains four shifted angle brackets and goes as z^4. Thus the worst possible behavior of A_R is $O(z)$.

We conclude that any diagrams with $m_2, m_3 \in R$ fall off as $O(z^{-1})\,\frac{1}{z}\,O(z^1) \sim O(z^{-1})$ for large z.

Case II: $m_3 \in L, m_2 \in R$. [12] Since we chose ℓ non-adjacent to m_2, the R subamplitude must have $n_R \geq 4$ legs. Hence all diagrams in this class must be of type A (MHV $\times$ MHV).

We need to analyze the large z behavior of the angle-brackets relevant for the MHV subamplitudes. As $z \to \infty$ we find that the leading behavior of $|\hat{\tilde{\ell}}\rangle$ and $|\hat{\tilde{P}}_{1K}\rangle$ is given by

$$|\hat{\tilde{\ell}}\rangle \;=\; |\ell\rangle - \frac{\langle m_2|1+K|X]}{\langle 1 m_2\rangle[\ell X]}|1\rangle + O(z^{-1})\,, \tag{A.16}$$

$$|\hat{\tilde{P}}_{1K}\rangle \;=\; \frac{\langle 1\ell\rangle}{\langle 1 m_2\rangle}|m_2\rangle + O(z^{-1})\,. \tag{A.17}$$

Short calculations then yield the following large z behavior for the relevant angle brackets:

$$\langle m_2\,\hat{\tilde{\ell}}\rangle \;\sim\; O(1)\,, \qquad \langle \hat{\tilde{\ell}}\,\hat{\tilde{P}}_{1K}\rangle \;\sim\; O(1)\,, \qquad \langle a\,\hat{\tilde{P}}_{1K}\rangle \;\sim\; O(1) \qquad \text{for any } a \notin \{m_2, \hat{\tilde{\ell}}\}\,, \tag{A.18}$$

but

$$\langle m_2\,\hat{\tilde{P}}_{1K}\rangle \;\sim\; O(z^{-1})\,. \tag{A.19}$$

To derive these falloffs, we used

$$\langle m_2|1+K+\ell|X] \neq 0\,, \tag{A.20}$$

which holds because the R subamplitude has more than 3 legs, as noted above.

Now consider the effect of the secondary shift on the MHV $\times$ MHV diagram:

- A_L: it follows from (A.18) that all angle brackets in the L subamplitude are $O(1)$, so $A_L \sim O(1)$.
- The propagator gives a factor of $1/z$.
- A_R: all angle brackets are $O(1)$, except for $\langle m_2 \hat{\tilde{P}}_{1K}\rangle$ which is $O(z^{-1})$ according to (A.19). Note that on the L MHV subamplitude, the internal line $\tilde{P}_{1K}$ *cannot have the common* SU(4) *index of* m_2 *and* m_3, because this index is already carried by lines 1 and m_3. Therefore, $\tilde{P}_{1K}$ on the R subamplitude *must have* this index in common with m_2. The "spin factor" in the numerator of the MHV subamplitude A_R thus includes at least one factor of $\langle m_2 \tilde{P}_{1K}\rangle$. If lines m_2 and $\tilde{P}_{1K}$ are non-adjacent in the R subamplitude then all angle brackets in the denominator are $O(1)$ according

[12] Note that the case of $m_2 \in L, m_3 \in R$ is obtained from this one by taking $m_2 \leftrightarrow m_3$ and $z \leftrightarrow -z$ in all expressions.

to (A.18). On the other hand, if lines m_2 and $\tilde{P}_{1K}$ are adjacent the denominator of A_R also contains one factor of $\langle m_2 \tilde{P}_{1K} \rangle$ and is thus $O(z^{-1})$. We conclude that at worst $A_R \sim O(1)$. Note that the common index of lines m_2 and m_3 was crucial to draw this conclusion.

We conclude that any diagrams with $m_3 \in L, m_2 \in R$ falls off at least as $O(1)\,\frac{1}{z}\,O(1) \sim O(z^{-1})$ for large z. The same argument holds also for the case $m_2 \in L, m_3 \in R$.

Case III: $m_2, m_3 \in L$. As the three lines 1, m_2 and m_3 all share a common SU(4) index, the L subamplitude must be NMHV in order to be non-vanishing. Thus there can be no MHV $\times$ MHV diagrams in this class. All amplitudes must be of type B (NMHV $\times$ anti-MHV). The right subamplitude is anti-MHV and must have $n_R = 3$ legs in order for the diagram to be non-vanishing. The secondary shift acts on the L subamplitude as a 3-line shift $[\tilde{1}, m_2, m_3|$. As particle 1 is a negative helicity gluon and as lines m_2 and m_3 share at least one common SU(4) index, the shift is precisely of the same type as the original secondary 3-line shift. This shift acts *only* on the L subamplitude, which has $n-1$ legs.

- A_L: the L subamplitude A_L goes as $1/z$ *provided* a $[1, m_2, m_3|$-shift with line 1 a negative helicity gluon and lines m_2 and m_3 sharing a common SU(4) index is a good shift for amplitudes with $n-1$ legs.
- The propagator is unshifted and thus $O(1)$.
- A_R: the right subamplitude is unshifted and thus $O(1)$.

We conclude that any diagrams with $m_2, m_3 \in L$ fall off as as $O(z^{-1})\,O(1)\,O(1) \sim O(z^{-1})$ for large z, assuming the validity of the same type of shift for $n-1$ legs.

In summary, the diagrams resulting from the $|1^-\ell\rangle$ vertex expansion of the amplitude A_n all fall off at least as fast as $1/z$ under the secondary shift $[1^-, m_2, m_3|$. For Case III, we needed to assume that a 3-line shift of type $[1^-, m_2, m_3|$ gives at least a falloff of $1/z$ for $(n-1)$-point amplitudes of the same type. We can thus use a simple inductive argument to show the validity of the shift for all $n \geq 8$. We will afterwards explicitly prove the falloff for $n = 6, 7$.

A.3 Induction

Let us assume that for some $n \geq 7$, all $\mathcal{N} = 4$ SYM NMHV n-point amplitudes A_n satisfy

$$A_n(1, x_2, \ldots, x_n) \to O(z^{-1}) \tag{A.21}$$

under any $[1, m_2, m_3|$-shift with line 1 a negative helicity gluon and lines $m_{2,3}$ sharing at least one common SU(4) index. Consider now a $[1, m_2', m_3'|$-shift on $A_{n+1}(1, x_2', \ldots, x_{n+1}')$, again with $m_{2,3}'$ chosen to share at least one common SU(4) index. We have shown above that ℓ can always be chosen such that all MHV $\times$ MHV vertex diagrams of the $[1^-, \ell\rangle$-shift fall off at least as $1/z$ under the $[1, m_2', m_3'|$-shift. Under the assumption (A.21), we have shown that NMHV$_n$ $\times$ anti-MHV$_3$ vertex diagrams will also fall off at least as $1/z$.

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For the cases $n = 6, 7$, our inductive step is not applicable to all diagrams because we *cannot* always pick line ℓ non-adjacent to m_2 and m_3. The diagrams where we cannot pick ℓ in this way must be analyzed separately. For $n = 7$, our reasoning above only fails for a small class of diagrams. Let us analyze this class of diagrams next.

A.4 Special diagrams for $n = 7$

For 7-point amplitudes there is one color ordering of the three lines 1, m_2 and m_3 which needs to be analyzed separately, namely

$$A_7(1, x_2, m_2, x_4, x_5, m_3, x_7)\,. \tag{A.22}$$

In this case we cannot choose ℓ to be non-adjacent to both m_2 and m_3. Instead choose $\ell = x_2$. The analysis of all diagrams goes through as in section A.2, except that Case II may now include a diagram of Type B (NMHV $\times$ anti-MHV), namely

$$A_L(\tilde{1}, -\tilde{P}_{1K}, x_4, x_5, m_3, x_7)\frac{1}{P_{1K}^2}A_R(\tilde{\ell}, m_2, \tilde{P}_{1K})\,. \tag{A.23}$$

It appears because ℓ is adjacent to m_2.

As $z \to \infty$ we find

$$|\hat{\tilde{1}}] = |\tilde{1}] + z\langle m_2 m_3\rangle|X]\,, \tag{A.24}$$

$$|\hat{\tilde{P}}_{1K}] = |\tilde{P}_{1K}] - z\,\frac{\langle 1 m_2\rangle\langle m_3 1\rangle}{\langle 1\ell\rangle}|X]\,, \tag{A.25}$$

$$|\hat{m}_3] = |m_3] + z\langle 1 m_2\rangle|X]\,, \tag{A.26}$$

while $|\tilde{P}_{1K}\rangle$ remains unshifted. Short calculations then yield the following large z behavior for the relevant square brackets:

$$[\hat{\tilde{1}}\hat{\tilde{P}}_{1K}] \sim O(z)\,, \qquad [a\hat{\tilde{P}}_{1K}] \sim O(z)\,, \qquad [a\hat{\tilde{1}}] \sim O(z) \qquad \text{for any } a \notin \{\hat{\tilde{1}}, \hat{\tilde{P}}_{1K}\}\,. \tag{A.27}$$

Now consider the effect of the secondary shift on the NMHV $\times$ anti-MHV diagram:

- A_L: after a rescaling $|\tilde{P}_{1K}\rangle \to -\frac{\langle 1 m_2\rangle}{\langle 1\ell\rangle}|\tilde{P}_{1K}\rangle$ and $|\tilde{P}_{1K}] \to -\frac{\langle 1\ell\rangle}{\langle 1 m_2\rangle}|\tilde{P}_{1K}]$, the shift acts exactly as a 3-line $[\tilde{1}, \tilde{P}_{1K}, m_3|$-shift. Note that line $\tilde{P}_{1K}$ on the L side has at least one common index with m_3, because line $\tilde{P}_{1K}$ cannot carry this index on the R side. In fact, this index is already carried by line m_2 in the right subamplitude, and as the right subamplitude is a 3-point anti-MHV amplitude, each index must occur exactly once for a non-vanishing result. The behavior of the L subamplitude is thus given by the falloff of a $n = 6$ amplitude under a $[\tilde{1}, \tilde{P}_{1K}, m_3|$ shift, in which line 1 is a negative helicity gluon and lines $\tilde{P}_{1K}$ and m_3 share a common index.

- The propagator gives a factor of $1/z$.

- A_R: the right subamplitude is a 3-point anti-MHV and is thus the ratio of four square brackets in the numerator and three square brackets in the denominator. According to (A.27) all square brackets are $O(z)$ and we conclude that $A_R \sim O(z)$ for large z.

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Provided $n = 6$ amplitudes fall off at least as $1/z$ under any 3-line shift $[1, m_2', m_3'|$ in which 1 is a negative helicity gluon and m_2' and m_3' share a common SU(4) index, we conclude that the full amplitude (A.22) goes as $O(z^{-1})\,O(z^{-1})\,O(z) \sim O(z^{-1})$ for large z.

We have thus reduced the validity of our shift at $n = 7$ to its validity at $n = 6$. Let us now analyze 6-point amplitudes.

A.5 Proof for $n = 6$

Our analysis above for $n > 6$ only used the fact that ℓ was non-adjacent to $m_{2,3}$ by ruling out certain diagrams of type B (NMHV $\times$ anti-MHV). For $n = 6$, we cannot rule out these diagrams and will thus analyze them individually below. Also, we will estimate the large z behavior of the NMHV=anti-MHV 5-point subamplitudes that appear in the $[1^-, \ell\rangle$-shift expansion. This will complete the explicit proof of the desired large z falloff at $n = 6$, without relying on a further inductive step.

Choose a $[1, m_2, m_3|$-shift where m_2 and m_3 share a common SU(4) index. Using the freedom to reverse the ordering of the states $123456 \to 165432$, there are six independent cases determined by the color ordering:

(a) $A_6(1, x_2, x_3, x_4, m_2, m_3)$

(b) $A_6(1, x_2, x_3, m_2, m_3, x_6)$

(c) $A_6(1, x_2, x_3, m_2, x_5, m_3)$

(d) $A_6(1, m_2, x_3, x_4, x_5, m_3)$

(e) $A_6(1, x_2, m_2, x_4, x_5, m_3)$

(f) $A_6(1, x_2, m_2, x_4, m_3, x_6)$

- Consider first the four cases (a)–(d). In these amplitudes, ℓ can be chosen to be non-adjacent to m_2,m_3. We pick

$$\text{(a),(b),(c):} \quad \ell \to x_2\,, \qquad \text{(d):} \quad \ell \to x_4\,. \tag{A.28}$$

  In all four situations the NMHV diagram is a Case III diagram ($m_2, m_3 \in L$), so we have to check the large z behavior of the Left 5-point anti-MHV amplitude under the $[1, m_2, m_3|$-shift. The denominator of the anti-MHV subamplitude will go as z^4 or z^5, depending on whether the lines $1, m_2, m_3$ are consecutive or not. The numerator contains four square brackets, at least one of which does not shift under $[1, m_2, m_3|$. This can be seen as follows. As the lines $1, m_2, m_3$ share a common SU(4) index, the other two lines in the 5-point amplitude, $\tilde{P}_{1K}$ and (say) y are the lines which do *not* carry this index. Since the numerator of an anti-MHV amplitude contains precisely the square brackets of particles which do not carry a certain SU(4) index, we conclude that there must be a factor of $[y\tilde{P}_{1K}]$ in the numerator.[13] This factor does

[13] To see this, one can also consider the conjugate MHV amplitude. Its numerator must contain a factor $\langle y\tilde{P}_{1K}\rangle$ because the conjugated particles on lines $\tilde{P}_{1K}$ and y share a common SU(4) index. Conjugating back, we replace angle by square brackets and obtain the factor $[y\tilde{P}_{1K}]$ in the numerator.

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not shift under the 3-line shift $[1, m_2, m_3|$, so the numerator grows as z^3 at worst. The 5-point anti-MHV L-subamplitude will thus have at least a $1/z$ falloff. As both the propagator and the right subamplitude remain unshifted, we conclude that the amplitudes (a)–(d) vanish at large z.

- Next, consider the amplitude (e) above: $A_6(1, x_2, m_2, x_4, x_5, m_3)$. Choose $\ell = x_4$. There are potentially two NMHV vertex diagrams: first, the (ℓ, x_5) channel which has $m_2, m_3 \in L$ and we can thus apply the same argument we used for cases (a)–(d). The diagram of this channel therefore falls off at least as $1/z$. Secondly, consider the (m_2, ℓ)-channel. This diagram has the same right subamplitude that we encountered for $n = 7$ in the diagram (A.23) above. By the same analysis we conclude that $A_R \sim O(z)$. Note that the left subamplitude is given by $A_L = A_5(\tilde{1}, x_2, \tilde{P}_{1K}, x_5, m_3)$. From eq. (A.27) we know that all square brackets involving $\tilde{1}, m_3, \tilde{P}_{1K}$ grow as $O(z)$, so the numerator will be at worst $O(z^4)$. As the three shifted lines $\tilde{1}, m_3, \tilde{P}_{1K}$ are not all consecutive, the denominator always goes as z^5. So $A_L \sim O(1/z)$, and as the propagator goes as $1/z$, we conclude that the whole diagram is at worst $O(1/z)$.

- Finally, consider case (f), $A_6(1, x_2, m_2, x_4, m_3, x_6)$. Choose $\ell = x_2$. Then the NMHV vertex appears in the diagram with channel (ℓ, m_2). This diagram can be treated just as the second diagram of case (e). To see this, note that $A_L = A_5(\tilde{1}, \tilde{P}_{1K}, x_4, m_3, x_6)$, so the three shifted lines, $\tilde{1}, \tilde{P}_{1K}, m_3$ are again not all consecutive. We conclude that this diagram also falls off at least as $1/z$ for large z.

We conclude that for a NMHV 6-point amplitude, any 3-line shift which involves a negative helicity gluon and two other states which share at least one common SU(4)-index, falls off at least as $1/z$ for large z. By the inductive argument of A.3 this immediately extends to all $n \geq 6$ and completes the proof.

B Anti-NMHV generating function from Anti-MHV vertex expansion

In section 6.1.2 we applied the Fourier transform prescription to obtain a generating function for anti-NMHV amplitudes,

$$\bar{\mathcal{F}}_n = \sum_I \bar{\mathcal{F}}_{n,I}\,, \qquad \bar{\mathcal{F}}_{n,I} = \frac{1}{\prod_{i=1}^n [i, i+1]}\, \overline{W}_I\, (\text{S.F.})_I\,. \tag{B.1}$$

In this appendix we use I to denote the diagrams of the anti-MHV vertex expansion. The sum is over all diagrams in any anti-MHV vertex expansion and the spin factor is

$$(\text{S.F.})_I = \frac{1}{2^4} \prod_{a=1}^4 \sum_{i,j=1}^n \sum_{k \in I_L} [i,j][kP_{I_L}]\, \partial_{ia}\partial_{ja}\partial_{ka}\, \eta_{1a} \cdots \eta_{na}\,. \tag{B.2}$$

The purpose of this appendix is to use the anti-MHV generating function to prove that (B.1)-(B.2) indeed is the correct generating function for an anti-MHV vertex diagram.

Consider any anti-MHV vertex diagram

$$A_{n_L}^{\text{anti-MHV}}(\dots)\frac{1}{P_I^2}A_{n_R}^{\text{anti-MHV}}(\dots)\,. \tag{B.3}$$

The value of each anti-MHV subamplitude is found by applying the appropriate derivative operators to the anti-MHV generating function, whose correctness we have already confirmed in section 6.1.1. The internal line must be an SU(4) invariant, so its total order is 4. Given the external states, there is a unique choice of internal state, so the 4 internal line differentiations can be taken outside the product of anti-MHV generating functions. Thus the value[14] of the diagram is

$$D_{\text{ext}}\; D_I\; \bar{F}_{n_L}(L)\; \bar{F}_{n_R}(R)\,. \tag{B.4}$$

This is true for any external states, hence the correct value of any anti-MHV vertex diagram is produced by the generating function

$$\begin{aligned}\bar{\mathcal{F}}_{n,I} &= D_I\bar{F}_{n_L}(L)\bar{F}_{n_R}(R) = D_I\left(\frac{1}{\prod_{i\in L}[i,i+1]}\widetilde{\delta^{(8)}(L)}\frac{1}{\prod_{j\in R}[j,j+1]}\widetilde{\delta^{(8)}(R)}\right)\\ &= \frac{1}{\prod_{i=1}^{n}[i,i+1]}\overline{W}_I\times(\text{S.F.})_I\,,\end{aligned} \tag{B.5}$$

with

$$(\text{S.F.})_I \;=\; D_I\,\widetilde{\delta^{(8)}(L)}\;\widetilde{\delta^{(8)}(R)}\,. \tag{B.6}$$

We have introduced

$$\widetilde{\delta^{(8)}(L)} \;\equiv\; \frac{1}{2^4}\prod_a\sum_{i,j\in L}[ij]\,\partial_i^a\partial_j^a\prod_{k\in L}\eta_{ka}\,, \tag{B.7}$$

and similarly for $\widetilde{\delta^{(8)}(R)}$.

The prefactors of (B.1) and (B.5) are clearly the same, so we just need to prove that the spin factor in (B.6) is equal to that in (B.2). We start from (B.6) and write out the full expressions for the "anti-delta-functions" (B.7), seperating out the internal line I from the external lines,

$$\begin{aligned}(\text{S.F.})_I \;=\; D_I\Bigg\{\prod_a\left[\sum_{i_L<j_L}[i_Lj_L]\,\partial_{i_L}^a\partial_{j_L}^a+\sum_{i_L}[i_LP_I]\,\partial_{i_L}^a\partial_I^a\right]\left(\prod_{k_L}\eta_{k_La}\right)\eta_{Ia}\\ \times\left[\sum_{i_R<j_R}[i_Rj_R]\,\partial_{i_R}^a\partial_{j_R}^a+\sum_{i_R}[i_RP_I]\,\partial_{i_R}^a\partial_I^a\right]\left(\eta_{Ia}\prod_{k_R}\eta_{k_Ra}\right)\Bigg\}\,.\end{aligned} \tag{B.8}$$

All lines $i_{L/R}, j_{L/R}, k_{L/R}$ are *external* states on the L/R side of the vertex expansion. Evaluate first the derivatives ∂_I inside the curly brackets to get

$$\begin{aligned}(\text{S.F.})_I \;=\; D_I\left[\sum_{i_L<j_L}[i_Lj_L]\,\partial_{i_L}^a\partial_{j_L}^a\left(\prod_{k_L}\eta_{k_La}\right)\eta_{Ia}+(-)^{n_L-1}\sum_{i_L}[i_LP_I]\,\partial_{i_L}^a\prod_{k_L}\eta_{k_La}\right]\\ \times\left[\sum_{i_R<j_R}[i_Rj_R]\,\partial_{i_R}^a\partial_{j_R}^a\,\eta_{Ia}\prod_{k_R}\eta_{k_Ra}+\sum_{i_R}[i_RP_I]\,\partial_{i_R}^a\prod_{k_R}\eta_{k_La}\right]\,.\end{aligned} \tag{B.9}$$

[14] It was described in [11] how to obtain the correct overall sign for the diagram.

Then perform the D_I differentiation:

$$(\text{S.F.})_I = \prod_a \Bigg[(-)^{n_L-1} \sum_{i_L<j_L} [i_L j_L]\, \partial^a_{i_L} \partial^a_{j_L} \prod_{k_L} \eta_{k_L a} \sum_{i_R} [i_R P_I]\, \partial^a_{i_R} \prod_{k_R} \eta_{k_L a} + (-)^{2n_L-1} \sum_{i_L} [i_L P_I]\, \partial^a_{i_L} \prod_{k_L} \eta_{k_L a} \sum_{i_R<j_R} [i_R j_R]\, \partial^a_{i_R} \partial^a_{j_R} \prod_{k_R} \eta_{k_R a} \Bigg] . \tag{B.10}$$

Now factor out the product of all η's corresponding to the external lines to find

$$(\text{S.F.})_I = \frac{1}{2^4} \prod_a \left[\sum_{i_L,j_L,i_R} [i_L j_L][i_R P_I] \partial^a_{i_L} \partial^a_{j_L} \partial^a_{i_R} - \sum_{i_R,j_R,i_L} [i_R j_R][i_L P_I] \partial^a_{i_R} \partial^a_{j_R} \partial^a_{i_L} \right] \prod_{\text{ext } k} \eta_{ka} \,. \tag{B.11}$$

Note that by the Schouten identity the antisymmetrized sum over 3 square brackets vanishes,

$$\sum_{i_R,j_R,k_R} [i_R j_R][k_R|\, \partial^a_{i_R} \partial^a_{j_R} \partial^a_{k_R} = 0 \,. \tag{B.12}$$

We can thus remove the L restriction on the index i_L in the second term in (B.11) and replace it by an index m running over all external states. We then obtain

$$\begin{aligned} \sum_{i_R,j_R,i_L} [i_R j_R][i_L P_I] \partial^a_{i_R} \partial^a_{j_R} \partial^a_{i_L} &= \sum_{i_R,j_R,m} [i_R j_R][m P_I] \partial^a_{i_R} \partial^a_{j_R} \partial^a_m \\ &= -\sum_{m,i_R,j_R} [m i_R][j_R P_I] \partial^a_m \partial^a_{i_R} \partial^a_{j_R} - \sum_{i_R,m,j_R} [j_R m][i_R P_I] \partial^a_{j_R} \partial^a_m \partial^a_{i_R} \,. \end{aligned} \tag{B.13}$$

In the second line we have used the Schouten identity to split the sum. This is done in order to complete the sum over i_L, j_L in the first term of (B.11) to a sum over all external states m and n

$$\begin{aligned} (\text{S.F.})_I &= \frac{1}{2^4} \prod_a \Bigg[\sum_{i_L,j_L,i_R} [i_L j_L][i_R P_I] \partial^a_{i_L} \partial^a_{j_L} \partial^a_{i_R} \\ &\qquad + \sum_{m,i_R,j_R} [m i_R][j_R P_I] \partial^a_m \partial^a_{i_R} \partial^a_{j_R} + \sum_{i_R,m,j_R} [j_R m][i_R P_I] \partial^a_{j_R} \partial^a_m \partial^a_{i_R} \Bigg] \prod_{\text{ext } k} \eta_{ka} \\ &= \frac{1}{2^4} \prod_a \left[\sum_{m,n,i_R} [mn][i_R P_I]\, \partial^a_m \partial^a_n \partial^a_{i_R} \right] \prod_{\text{ext } k} \eta_{ka} \,. \end{aligned} \tag{B.14}$$

We had to again use (B.12) in the last step. Finally the Schouten identity allows us to convert the sum over external momenta on the R to a sum over external momenta on the L subamplitude. The result is

$$(\text{S.F.})_I = \frac{1}{2^4} \prod_a \left[\sum_{m,n,i_L} [mn][i_L P_I]\, \partial^a_m \partial^a_n \partial^a_{i_L} \right] \prod_{\text{ext } k} \eta_{ka} \,. \tag{B.15}$$

This is precisely the expected spin factor (B.2) (given in the main text in (6.14)) that our prescription predicts.

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